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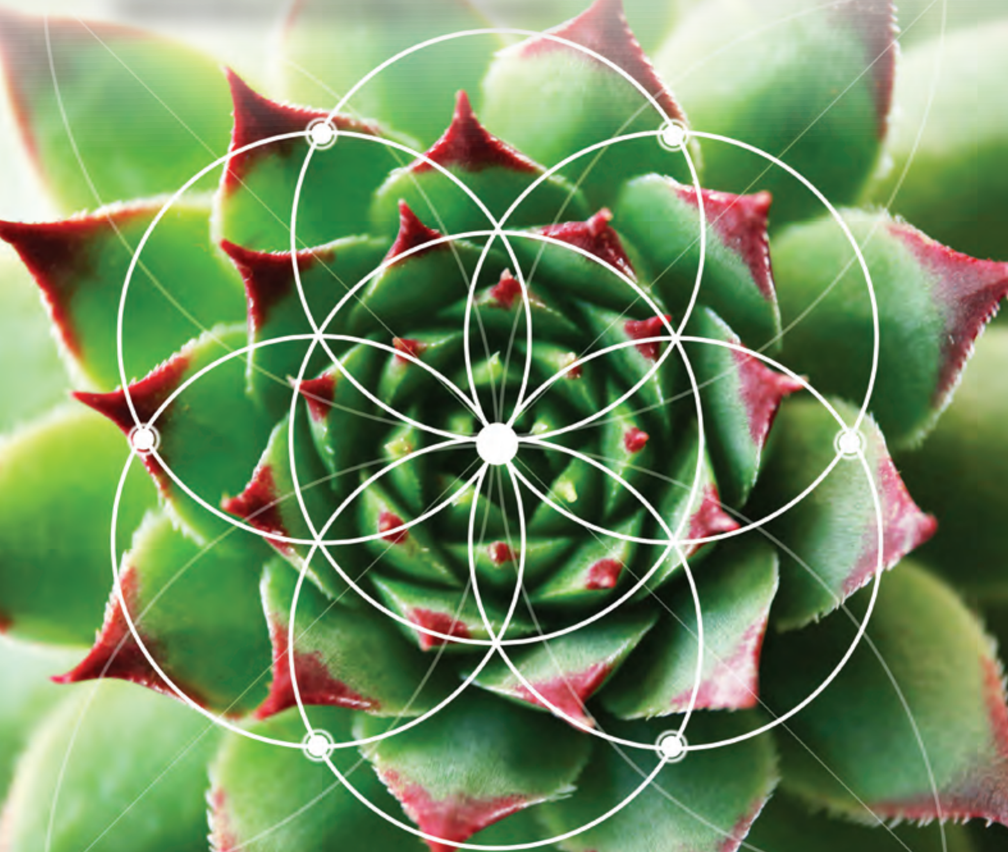
Topic
Science & Mathematics

Subtopic
Mathematics

The Power of Mathematical Visualization

Course Guidebook

Professor James S. Tanton
Mathematical Association of America



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James S. Tanton is the Mathematician-at-Large for the Mathematical Association of America (MAA) in Washington DC. He received his Ph.D. in Mathematics from Princeton University in 1994. From 2004 to 2012, he worked as a full-time high school teacher at St. Mark's School in Southborough, Massachusetts. In 2004, Dr. Tanton founded the St. Mark's Institute of Mathematics, an outreach program promoting joyful and effective mathematics education for both students and educators.

Believing that mathematics really is accessible to all, Dr. Tanton is committed to sharing the delight and beauty of the subject. He is actively engaged in professional development for educators in the United States, in Canada, and overseas.

Dr. Tanton is the author of *Solve This: Math Activities for Students and Clubs*; *The Encyclopedia of Mathematics*; *Mathematics Galore!*; *Trigonometry: A Clever Study Guide*; *Without Words: Mathematical Puzzles to Confound and Delight*; *More Without Words*; *8 Tips to Conquer Any Problem*; and 12 self-published texts. He received the 2004 Beckenbach Book Prize from the MAA for *Solve This*, the 2006 George Howell Kidder Faculty Prize from St. Mark's School, and a 2010 Raytheon Math Hero Award for excellence in math teaching. He also publishes research and expository articles, and his extracurricular classes have helped high school students pursue research projects and publish their results.

Dr. Tanton's other Great Course is entitled *Geometry: An Interactive Journey to Mastery*. ■

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THE POWER OF MATHEMATICAL VISUALIZATION

COURSE SCOPE

Too many people believe that they simply aren't good at math.

But what if you could learn to see mathematical problems in new and helpful ways? What if you could return to difficult mathematics topics from your past with renewed power, ease, and joy? The art of thinking visually is a key component to mathematical success.

They say that a picture can speak a thousand words. In mathematics, a picture can spawn a thousand ideas. A picture can provide deep understanding. It can prompt that aha moment to cause an idea or process to suddenly make sense and thus lead the way to finally understanding a tricky piece of mathematics.

This course offers you an alternative entry into the topic of mathematics. It provides a natural way to think about ideas and truly see what they are about and how to make them work for you. Visualization is regularly employed by mathematicians to make brilliant insights, forge new paths of discovery, and find deeper understanding in the world of mathematics. And it isn't a skill merely for the mathematically inclined. You, too, can acquire the tools necessary to think brilliantly in mathematics, experience tremendous success in learning and analyzing mathematical ideas, and achieve—finally—deep and powerful understanding of the mathematical concepts you encountered in school and use today.

The goal of this course is to reveal how mathematics, if seen the right way, can become robust, complete, and pleasing. And from this, a wonderful sense of ownership of ideas emerges. Mathematics is transformed from an experience of hazy thinking and memorization to one of powerful clarity, insight, and fun!

The course begins with the joyous play of numbers, and you will discover astounding and deep surprises from simple counting pictures. (You will learn, for example, how to add all the numbers from 1 up to 1,000 and back down again in less than a second!)

You will make sense of negative numbers once and for all and tackle some age-old word problems from school, all with the ease of pictures to help. You will also answer this troublesome question: Why is negative times negative positive?

Next, you will explore the mathematics of fractions and decimals in a new and utterly intuitive manner, and you will find that there are numbers that cannot be expressed as fractions, the irrational numbers. This will segue into some astounding conceptual mathematics: the world of mathematical infinities.

The second half of the course turns to a sample of individual topics, some familiar from high school mathematics and some that might be new to you. You will explore the mathematics of probability, probe Pascal's triangle, and discover its connection to random motion. You will examine the mathematics of the Fibonacci numbers, fixed points, and folding patterns. Some traditional high school algebra will be revitalized, and you will consider ideas from statistics in new, enlightening ways. And you will accomplish all of this by using the joy and power of visual thinking.

This course demonstrates that every mathematical topic—whether it is a study of simple counting or subtle probability theory—serves as a portal to further thought and wonder. A universe of joyous mathematics is yours to behold!

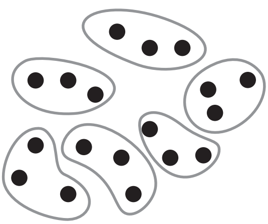
THE POWER OF A MATHEMATICAL PICTURE

LECTURE 1

If a picture is worth a thousand words, in mathematics, a picture can spawn a thousand ideas. A picture can provide deep understanding, prompt an idea or process to suddenly make sense, and lead the way to finally understanding a tricky piece of mathematics. The focus of this lecture is on sums. You will learn how to quickly add all the numbers up to 1000 and back down, learn about sums of odd numbers and of even numbers, and even establish Galileo's results on ratios of sums of numbers—all through the use of a single picture.

BASIC ARITHMETIC

1.1



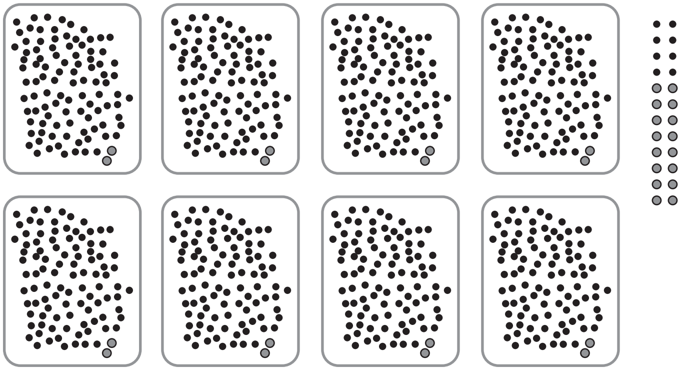
$18 \div 3 = 6$

We can circle groups of dots in pictures to make sense of division. For example, the division problem $18 \div 3$ is asking the following question: How many groups of 3 can you find in a picture of 18 dots? There are 6 of them, so $18 \div 3 = 6$.

We can push this visual picture further and make sense of some complicated division problems. For example, what is $808 \div 98$? We can see that the answer has to be 8 with a remainder of 24.

You can imagine looking for groups of 100, rather than 98. (The number is 98 is too difficult.) If we visualize this (**figure 1.2**), we see that there will be 8 of these groups, with 8 dots left over.

1.2



808 ÷ 98 = 8 r24

But each group of 100 is itself off by 2 dots—we wanted groups of 98—so we have an extra 16 dots floating around. That makes for 8 groups of 98 and a remainder of 16 and 8, which is equal to 24 dots. Therefore, 808 ÷ 98 = 8 with a remainder of 24.

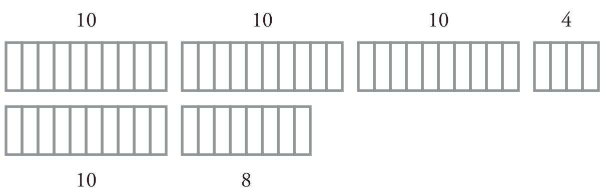
1.3

$$\begin{array}{r} {}^{2}\cancel{3}{}^{1}4 \\ -18 \\ \hline 16 \end{array}$$

When asked to do 34 – 18, we can certainly do the traditional algorithm and get the answer, 16.

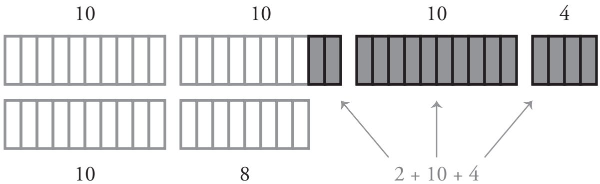
But can't we just see in our minds that the answer has to be 2 + 10 + 4, which is 16? Line up a row of 34 blocks and a row of 18 blocks side by side.

1.4



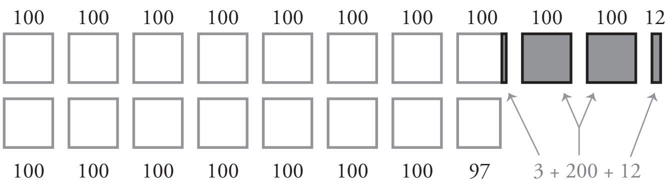
Now we can see that the 2 rows differ by 2 and 10 and 4 blocks, so the difference is 16.

1.5



In the same way, 1012 – 797 has to be 3 and 200 and 12—which is 215. From 797 to 800 is 3, from 800 to 1000 is 200, and there is an extra 12, for a total of 215.

1.6



1.7

$$\begin{array}{r} 1005 \\ \underline{-387} \end{array}$$

This flexibility of thought helps with subtraction in general. For example, consider 1005 – 387.

We have a lot of borrowing to do if we follow the traditional approach: 5 – 7, 0 – 8, and 0 – 3 all need borrows.

But we can make this work simpler.

1.8

$$\begin{array}{r} 1000 \\ \underline{-382} \end{array}$$

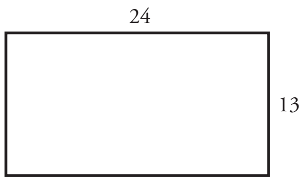
We are looking for the difference between 1005 dots and 387 dots. Let’s make 1005 friendlier and turn it into 1000. Remove 5 from each and just compute the difference between 1000 and 382 instead. Now we can see the answer: 8 + 10 + 600, or 618.

1.9

$$\begin{array}{r} 999 \\ \underline{-381} \end{array}$$

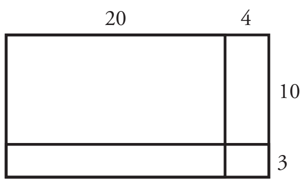
But if we still want to do the traditional algorithm, then we can remove 1 more dot from each pile and make the problem 999 – 381.

1.10



Now we can do the algorithm without any borrows: 9 – 1, 9 – 8, and 9 – 3. This way, we've made the problem much easier to do, even if someone insists that we use the algorithm.

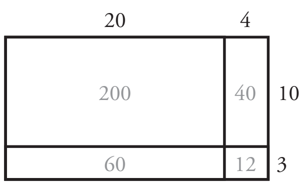
1.11



Isn't multiplication really a geometry problem? Isn't 24 × 13, for example, just asking for the area of a rectangle that is 24 units wide and 13 units high?

Then why not just chop up the rectangle into pieces that are manageable? For example, think of 24 as 20 and 4, and 13 as 10 and 3.

1.12



Then we see that 24 × 13 must be the areas of the individual pieces added together: 200 + 40 + 60 + 12 = 312.

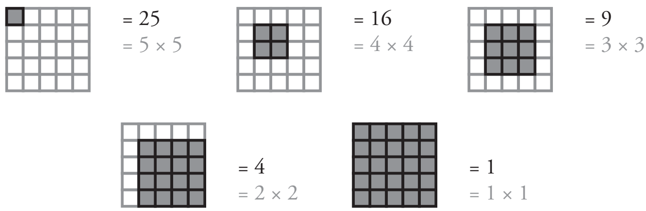
GRIDS

In a 5-by-5 grid of squares, there are 25 small 1×1 squares within the grid.

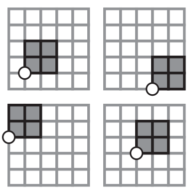
But we can count 2×2 squares as well. There are 16 of these in total. If we count the 3×3 squares, there turns out to be 9 of those. And there are 4 of the 4×4 squares. Finally, there is 1 large 5×5 square.

So, there are 25 1×1 squares, 16 2×2 squares, 9 3×3 squares, 4 4×4 squares, and 1 5×5 square. Each count of squares is itself a square number!

1.13

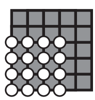


1.14



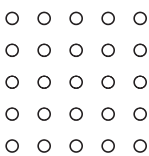
Why does counting squares on a square grid give square-number answers? Let's focus on the lower-left corners of the squares we're counting. For example, of the 2×2 squares, some possible lower-left corners can be seen in **figure 1.14**.

1.15



Let's draw all of the possible lower-left corners as shown in **figure 1.15**. Now we see that there is a square array of them, 4 × 4 of them, which is 16. Thus, there are 16 2×2 squares. This image makes it clear why the count of squares is always a square number.

1.16



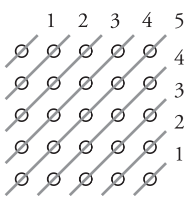
Let's view the 5-by-5 grid as an array of dots as in **figure 1.16**.

This is certainly a picture of 25 dots, but can you see in this picture the sum 1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 +1?

Look at the diagonals: 1, 2, 3, 4, 5, 4, 3, 2, 1.

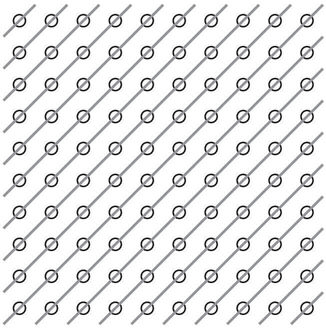
The sum we seek matches the diagonals of the square. There are 25 dots in all, so without doing any arithmetic, we can say that the value of the sum must be 25.

1.17



1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25

1.18



$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 +$
$9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 100$

What is the sum of all the numbers $1 + 2 + 3 + \ldots$ up to 10 and back down again?

This sum must come from the diagonals of a 10-by-10 array of dots.

Again, without any arithmetic, the value of the sum must be 10 squared (10^2): 100.

What is the sum of all the numbers from 1 to 1000 and back down again? It must be 1000 squared, from a 1000-by-1000 array of dots. That's 1 million.

If you were to compute this on a calculator—$1 + 2 + 3 + \ldots$—it would take forever. But the answer is available to us quickly via this picture.

$$1 + 2 + 3 + \ldots + 998 + 999 + 1000 + 999 + 998 + \ldots + 3 + 2 + 1 = 1000 \times 1000 = 1{,}000{,}000$$

THE SUM OF NUMBERS

There is a general formula for the sum of numbers.

$$1 + 2 + 3 + \ldots + n = \frac{n^2 + n}{2}$$

The sum of the first n numbers, $1 + 2 + 3$ all the way up to some number n, is $(n^2 + n) \div 2$. For example, the sum of the first 5 numbers, $1 + 2 + 3 + 4 + 5$, is $5^2 + 5 = 25 + 5 = 30$, and $30 \div 2 = 15$. And we can check that $1 + 2 + 3 + 4 + 5$ is indeed 15.

Where does this formula come from, and why is it true?

Our 5×5 array of dots gave us something akin to this result. We have that 1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25. Can we get from this answer to just 1 + 2 + 3 + 4 + 5?

If we look at what we have, we see that the sum we want, 1 + 2 + 3 + 4 + 5, is the left half of the equation.

$$1 + 2 + 3 + \ldots + n = \frac{n^2 + n}{2}$$

$$\boxed{1 + 2 + 3 + 4 + 5} + 4 + 3 + 2 + 1 = 25$$

Actually, half is not quite right. The right portion of the equation is missing a 5. It's just the sum 1 + 2 + 3 + 4. We want to see an additional 5, so let's add a 5 on the left—and to keep things balanced, we need to add a 5 to the right as well.

$$\boxed{1 + 2 + 3 + 4 + 5} + 5 + 4 + 3 + 2 + 1 = 25 + 5$$

Now we see 2 copies of what we want. Twice the sum we seek is 25 + 5. So, this means that the sum itself is half of this.

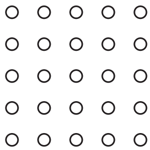
1 + 2 + 3 + 4 + 5 is indeed $(5^2 + 5) \div 2$. And this matches the general formula. There is nothing special about the number 5. The same ideas show that the sum of the first n counting numbers must be half of $n^2 + n$.

$$2 \times (1 + 2 + 3 + 4 + 5) = 25 + 5$$

$$1 + 2 + 3 + 4 + 5 = \frac{25 + 5}{2}$$

THE SUM OF ODD NUMBERS

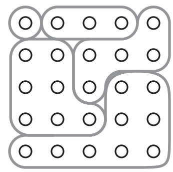
1.19



$1 + 3 + 5 + 7 + 9$

Look at the 5-by-5 grid of dots again. Do you see the sum $1 + 3 + 5 + 7 + 9$, the sum of the first 5 odd numbers?

1.20

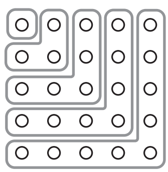


$1 + 3 + 5 + 7 + 9$

We can certainly circle these groups randomly and make them fit.

But such a random picture isn't enlightening. We want to see a picture that isn't locked into this particular example of 25 dots. We want a picture that speaks to a higher truth and clearly holds for all possible square arrays. Mathematicians are always on the lookout for this sort of thing, and symmetry is often a pointer to higher truths.

1.21

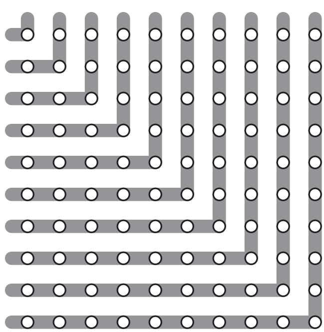


$1 + 3 + 5 + 7 + 9 = 25$

Do you see $1 + 3 + 5 + 7 + 9$ in the 5-by-5 array of dots in a way that speaks to a higher truth? Think L shapes.

The sum of the first 5 odd numbers is hidden in the 5-by-5 array as Ls. The sum $1 + 3 + 5 + 7 + 9$ must be 5^2, or 25.

1.22



In the same way, the sum of the first 10 odd numbers, $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$, sit in a 10-by-10 array of dots and therefore must have an answer of 100, the count of dots in that array.

In general, the sum of the first n odd numbers must be n^2.

$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 10^2 = 25$

Galileo, lived at the turn of the 16th century and is revered today for his work in science and mathematics, thought to make fractions out of the odd numbers. For example, take the first 5 odd numbers and use their sum for the numerator of a fraction and the sum of the next 5 odd numbers for its denominator. This gives a fraction that simplifies to ⅓.

$$\frac{1+3+5+7+9}{11+13+15+17+19} = \frac{25}{75} = \frac{1}{3}$$

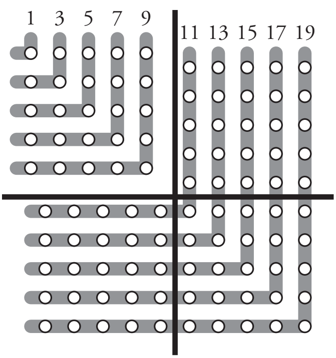
Do the same for the first 2 odd numbers, followed by the next 2. You get ⅓ again.

$$\frac{1+3}{5+7} = \frac{4}{12} = \frac{1}{3}$$

Do it again for the first 10 odd numbers, and the next 10. It's ⅓ again!

$$\frac{1+3+5+7+9+11+13+15+17+19}{21+23+25+27+29+31+33+35+37+39} = \frac{100}{300} = \frac{1}{3}$$

1.23



$$\frac{1+3+5+7+9}{11+13+15+17+19} = \frac{1}{3}$$

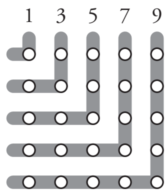
Galileo observed that all the fractions made out of the odd numbers this way are equal. They all equal ⅓. These fractions are today called the Galilean ratios.

There is a connection between the ratios and the L shapes in squares. **Figure 1.23** is purely visual proof of the Galilean ratios.

The first 5 L shapes, the sum of the first 5 odd numbers, makes 1 block of 25 dots. The next 5 L shapes for the next 5 odd numbers makes 3 blocks of 25 dots. So, the first 5 odd numbers make for ⅓ of the next 5 odd numbers.

THE SUM OF EVEN NUMBERS

1.24

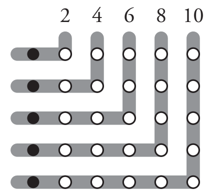


$1 + 3 + 5 + 7 + 9 = 25$

$2 + 4 + 6 + 8 + 10 = ?$

Are there results about sums of even numbers, too? For example, we have a picture for the sum of the first 5 odd numbers. Can we get from this a picture of the first 5 even numbers, $2 + 4 + 6 + 8 + 10$?

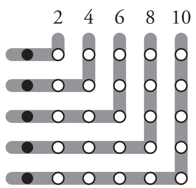
1.25



Just add a dot to each L shape!

This has turned the 5-by-5 square into a rectangle. The sum of the first 5 even numbers must be the 5×5 we had before plus 5 more, $5^2 + 5$, which is 30. (See **Figure 1.26**.)

1.26



$1 + 3 + 5 + 7 + 9 = 25$

$2 + 4 + 6 + 8 + 10 = \underbrace{25 + 5}_{5^2 + 5} = 30$

In general, the sum of the first n even numbers must come from the picture of n^2 dots plus an extra n dots: $n^2 + n$.

We're coming full circle, because we have seen the expression $n^2 + n$ before.

Take the sum of the first 5 even numbers. It equals $5^2 + 5$.

Now divide everything by 2: $2 \div 2$, $4 \div 2$, $6 \div 2$, $8 \div 2$, $10 \div 2$, and $(5^2 + 5) \div 2$.

$$\frac{2}{2} + \frac{4}{2} + \frac{6}{2} + \frac{8}{2} + \frac{10}{2} = \frac{5^2 + 5}{2}$$

$$1 + 2 + 3 + 4 + 5 = \frac{5^2 + 5}{2}$$

And we're back to the formula $1 + 2 + 3 + 4 + 5 = (5^2 + 5) \div 2$.

We have come full circle. We're back to the general formula for the sum of numbers.

FURTHER EXPLORATION

WEB

Tanton, "25 People in a Square Grid Puzzle."
http://www.jamestanton.com/?p=806
———, "The Sum 1+2+…+N."
http://www.jamestanton.com/?p=1006
———, "Sums of Cubes."
http://www.jamestanton.com/?p=820

READING

Conway and Guy, *The Book of Numbers.*
Nelson, *Proof without Words.*
———, *Proof without Words II.*
Tanton, *More without Words.*
———, *Thinking Mathematics! Vol. 1.*
———, *Without Words.*
Tanton, et al, "Young Students Explore Proofs without Words."

PROBLEMS

1. **a** What is the sum of the first 1000 counting numbers?

b What is the sum of the first 1000 odd numbers? (What is the thousandth odd number?)

c What is the sum of the first 1000 even numbers?

2. Draw a picture to show that the sum of the first 3 odd numbers must be $\frac{1}{8}$ the sum of the next 6 odd numbers.

SOLUTIONS

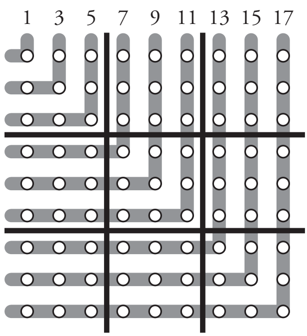
1. **a** $1 + 2 + \ldots + 1000 = \dfrac{1000^2 + 1000}{2} = 500{,}500.$

b The one-thousandth odd number is 1999 and the sum of the first 1000 odd numbers: $1 + 3 + 5 + \ldots + 1999$, is $1000^2 = 1{,}000{,}000$.

c The sum of the first 1000 even numbers: $2 + 4 + 6 + \ldots + 2000$, is $1000^2 + 1000 = 1{,}001{,}000$. (Divide this by 2 and get back to the sum of the first 1000 counting numbers!)

2. See **figure 1.27**.

1.27



In general, we have:

$$\frac{1}{3+5} = \frac{3+5}{7+9+11+13} = \frac{1+3+5}{7+9+11+13+17} = \frac{1}{8}.$$

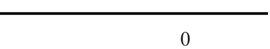
VISUALIZING NEGATIVE NUMBERS

LECTURE 2

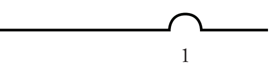
The positive counting numbers constitute only half of the integer world. In addition to these numbers, there are the negative numbers. Imagine the new heights we might reach by doubling the types of numbers we consider. But negative numbers can be tricky; working with them is subtle and difficult. In this lecture, you will learn to overcome their difficulties and make sense of them. Then, when you are ready, you will play with them.

SUBTRACTION: THE ADDITION OF THE OPPOSITE

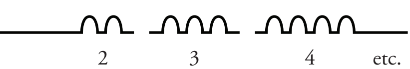
2.1



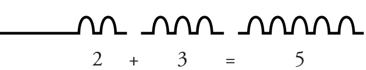
When James was a young child, his parents sat him in a sandbox in the backyard. Being a very tranquil child, he found it pleasing to start each session in his sandbox by leveling the sand to make a perfectly flat horizontal surface. Then, he would spend hours admiring the perfection of that level nothingness. His admiration for this levelness was so high that he decided to give this flat state a name: zero.



But then, one day, after hours of meditating on the wonders of nothingness, James had an epiphany. He realized that he could reach behind him, grab a handful of sand, and make a pile. And he called the pile one.

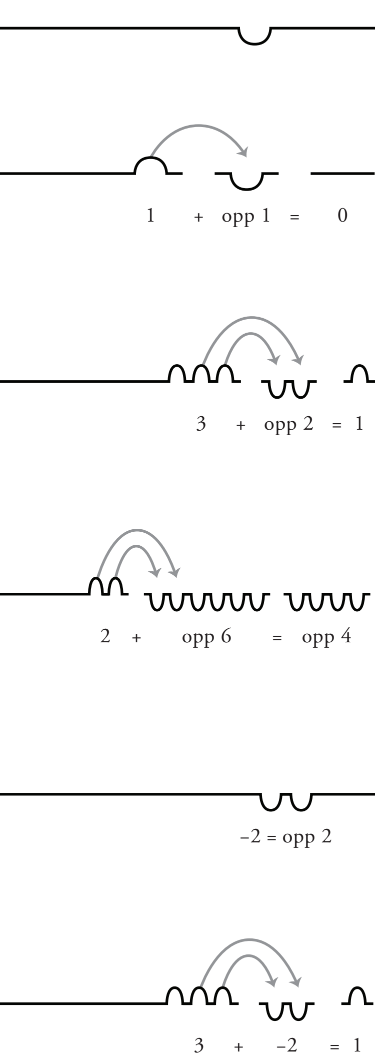


But he also realized that he could make 2 piles—a state he called two—3 piles, and 4, and so on. So, on this day, James discovered the positive counting numbers.



And he discovered arithmetic. For example, as a young child, he could see that 2 + 3 = 5 through piles of sand.

2.2



But then he had the most astounding epiphany of all: Instead of making piles, he realized that he could make holes.

He eventually decided to call a hole the opposite of a pile, because he realized that a hole and a pile when put together annihilate each other. He decided to denote a hole as "opp 1" for the opposite of a pile.

So, on this wondrous day, he discovered the equivalent of negative numbers. And he could do arithmetic with negative numbers, too. For example, he could see that 3 piles and 2 holes make a 1 pile: 3 + opp 2 = 1.

He saw that 2 piles and 6 holes make 4 holes: 2 + opp 6 = opp 4—and so on.

This piles and holes model is exceptionally straightforward. It is society that introduces the confusion. We as a society don't use the word "opposite" to denote opposite. We use a tiny dash and call it a negative sign. So, –2 really means the opposite of 2, or the opposite of 2 piles—that is, 2 holes.

And an equation like 3 + opp 2 = 1 is now written as 3 + –2 = 1.

There is an additional piece of confusion: People don't like a plus sign and a negative sign sitting next to each other. To avoid this occurrence, an operation called subtraction is introduced—the idea of taking away. Because placing 2 holes next to 3 piles has the same effect as taking away 2 of the piles, people often prefer to think of 3 + –2 as 3 take away 2 and write 3 – 2.

So, the addition of the opposite is couched as an operation of subtraction.

The idea of subtraction is introduced to our youngsters first in our society, before the idea of negative numbers, and it is difficult for adults to unthink subtraction. But all is easier if we can.

Just think of it like this: There is no such thing as subtraction. Subtraction is really the *addition* of the opposite. Write and think about everything in terms of opposites. In particular, realize that the negative sign itself, the little dash, just means opposite.

What is the opposite of 3 piles and 2 holes? That question is ambiguous. Depending on how we read it, it can mean either of the following: What is the opposite of 3 piles, and then 2 holes? *or* What is the opposite of … 3 piles and 2 holes?

We want to ask the question the second way.

Mathematicians use parentheses to clarify matters, to show when objects are meant to be grouped together. The following is our question in mathematical notation:

What is $-(3-2)$?

The opposite of 3 piles and 2 holes is 3 holes and 2 piles.

$$-(3-2) = -3 + 2$$

If we're interested in the actual answer, it's 1 hole.

What is the opposite of x piles and y holes?

The answer is clearly x holes and y piles. And when we write this out, it looks like we're suddenly in high school algebra class.

$$-(x-y) = -x + y$$

Think piles and holes even for algebra.

Here's a typical algebra textbook question:

Simplify $10 - (5 - x)$.

We have 10 piles, and the opposite of … 5 piles and x holes. That's 10 piles and 5 holes and x piles. 10 piles and 5 holes make 5 piles, so we have in the end 5 piles and x piles: $5 + x$.

$$10 - (5 - x) = 10 + -5 + x = 5 + x$$

We just handled, with natural ease, the bane of many high school and college algebra students: the act of distributing the negative sign. If you always think piles and holes, there is no issue. You always know how to handle a negative sign outside a set of parentheses.

Here's a question that would normally cause great confusion:

What is $-x$ if x is -7?

Think piles and holes. If x is -7, then x is 7 holes.

The questions is asking: What is $-x$—that is, what is the opposite of x—if x is 7 holes? The answer is 7 piles. So, what is $-x$ if x is -7? The answer is 7.

$$-x = 7$$

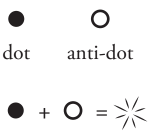
Most people think that $-x$ just has to give a negative answer because they see a negative sign. But think opposite and piles and holes, and all is clear: $-x$ will be positive if x is a collection of holes.

EXTENDING GALILEO'S RESULTS

In the previous lecture, we established the Galilean ratios through pictures of dots. Let's see if we can extend Galileo's results to the world of negative numbers.

In this lecture, we've been dealing with piles and holes, with holes being the opposite of piles. In the previous lecture, we dealt with dots. So, let's entertain the idea of having the opposite of a dot—an anti-dot.

2.3



Dots are solid circles and anti-dots are hollow circles. Like piles and holes, like matter and antimatter, when you put a dot and anti-dot together, they annihilate and leave you with the zero state.

2.4

In this work, dots represent positive numbers while anti-dots represent negative numbers, and putting dots and anti-dots together induces cancellations. **Figure 2.4** shows what 5 + –2 looks like in this dot/anti-dot world: We get 3 dots.

2.5

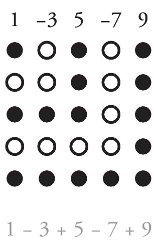
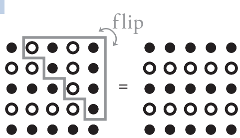


Figure 2.5 is a picture from the previous lecture, but this time we're seeing some dots as positive and others as negative. Do you see the sum 1 – 3 + 5 – 7 + 9? In other words, do you see 1 dot, 3 anti-dots, 5 dots, 7 anti-dots, and 9 dots appearing in the L shapes?

We can move dots around in this picture and it won't change the sum. Look at what happens if we just flip a corner of dots.

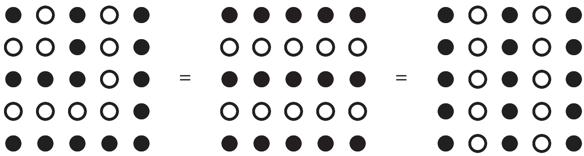
2.6



In the picture on the right, we can see how dots and anti-dots annihilate. There will be 5 actual dots surviving. The value of the sum must be 5. But let's not focus on that value; let's keep playing with the pictures.

We now have 2 pictorial ways to represent the same sum. We can add a third picture if we just rotate the second picture 90°. Each picture has the same number of black dots and white dots. They are just rearrangements of the Ls in the original square.

2.7



2.8

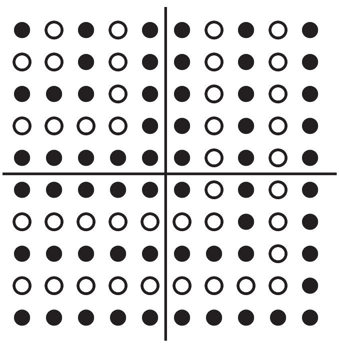


Figure 2.8 is the full picture from the previous lecture, the one that proves that 1 + 3 + 5 + 7 + 9 is ⅓ of 11 + 13 + 15 + 17 + 19, the sum of the next 5 odd numbers.

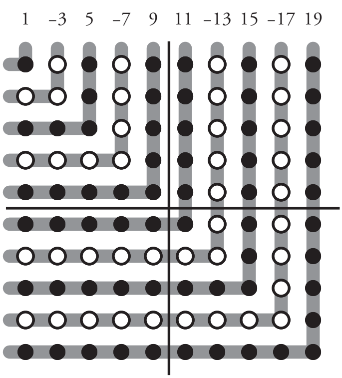
But in this lecture, we're looking at solid dots as positive numbers and empty dots as negative numbers. With this new way of looking at it, what is the picture telling us?

The top-left square is now the sum 1 – 3 + 5 – 7 + 9. The remaining 3 squares, looking at the L shapes, make 11 – 13 + 15 – 17 + 19.

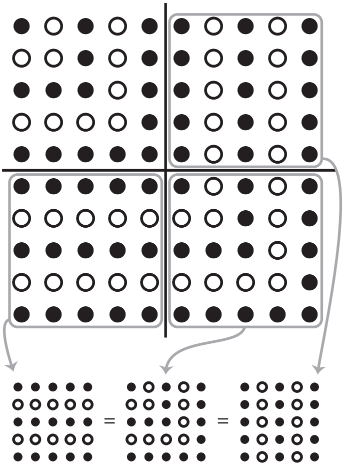
But the remaining 3 squares are also just rearranged copies of the top-left square. (See **figure 2.10**)

So, the top-left square represents ⅓ of the remainder of the big square—that is, the sum 1 – 3 + 5 – 7 + 9 must be ⅓ of 11 – 13 + 15 – 19 + 21. The alternating

2.9



2.10



sum of the first 5 odd numbers divided by the alternating sum of the next 5 odd numbers is ⅓.

$$\frac{1 - 3 + 5 - 7 + 9}{11 - 13 + 15 - 17 + 19} = \frac{1}{3}$$

This is a new variant of the Galilean ratio, one with negative entries. Previously, we had the ratio with all numbers positive.

$$\frac{1 + 3 + 5 + 7 + 9}{11 + 13 + 15 + 17 + 19} = \frac{1}{3}$$

We can do this for other-sized dot grids, too, and get lots of ratios like these. In fact, we can take a whole slew of Galileo's ratios, insert some negative signs, and change nothing.

Galileo without negative numbers:

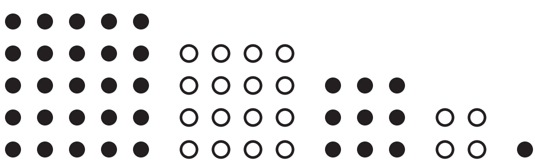
$$\frac{1}{3} = \frac{1+3+5}{7+9+11} = \frac{1+3+5+7+9}{11+13+15+17+19} = \frac{1+3+5+7+9+11+13}{15+17+19+21+23+25+27} = \ldots$$

Today with negative numbers:

$$\frac{1}{3} = \frac{1-3+5}{7-9+11} = \frac{1-3+5-7+9}{11-13+15-17+19} = \frac{1-3+5-7+9-11+13}{15-17+19-21+23-25+27} = \ldots$$

Draw a 5-by-5 array of dots, a 4-by-4 array of anti-dots, a 3-by-3 array of dots, a 2-by-2 array of anti-dots, and a 1-by-1 dot.

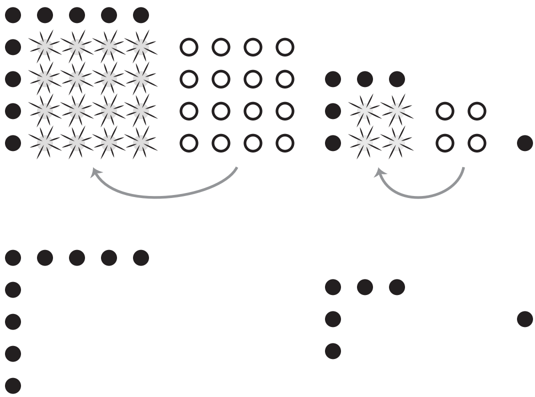
2.11



Now put the 4-by-4 array on top of the 5-by-5 array. A lot of annihilations occur, leaving an L shape of dots.

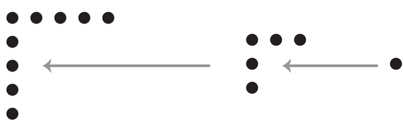
Similarly, putting the 2-by-2 array on the 3-by-3 picture also leaves an L shape of dots. And we still have the single dot. It is its own tiny L shape.

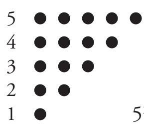
2.12



We have 3 L shapes. Push them together to make a triangle. We can see that this triangle represents the sum 1 + 2 + 3 + 4 + 5.

2.13





$$5^2 - 4^2 + 3^2 - 2^2 + 1^2 = 1 + 2 + 3 + 4 + 5$$

$$n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 \ldots + 3^2 - 2^2 + 1^2 = 1 + 2 + 3 + \ldots + n$$

(for n odd)

We have just proved the curious result that
$5^2 - 4^2 + 3^2 - 2^2 + 1^1 = 1 + 2 + 3 + 4 + 5.$

In general, this idea establishes that the alternating sum of the first *n* square numbers will always equal the sum of the first *n* counting numbers, at least if we start with an odd square.

FURTHER EXPLORATION

WEB

Tanton, "An Aside on Negative Numbers."
http://gdaymath.com/lessons/powerarea/1-4-an-aside-on-negative-numbers-piles-and-holes/

READING

Nelson, *Proof without Words*.
———, *Proof without Words II*.

PROBLEMS

1. Draw a picture to show that $4^2 - 3^2 + 2^2 - 1^2 = 1 + 2 + 3 + 4$.

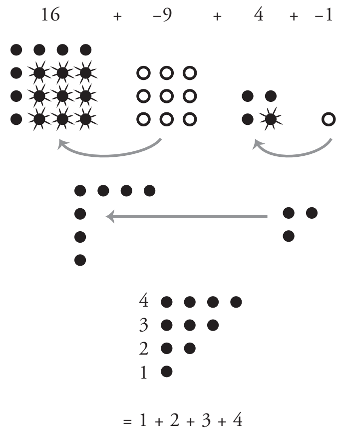
2. Cecile shuffles together 17 red cards and 21 blue cards. She then splits the pile of 38 cards into a small pile of 10 cards and a large pile of 28 cards. She counts the number of red cards in the small pile and the number of blue cards in the large pile. Explain why these 2 counts must differ by exactly 11.

3. Is −0 equal to 0?

4. Is 0 even or odd? Is −6 even or odd? Is −17 even or odd?

SOLUTIONS

1. 2.14



In general, the following formula holds for both n odd and n even:

$$n^2 - (n-1)^2 + (n-2)^2 - \ldots \pm 1^2 = 1 + 2 + 3 + \ldots + n.$$

2. Let R be the count of red cards in the small pile.

2.15

	Small pile of 10	Large pile of 28
17 red cards	R	$17 - R$
21 blue cards		$28 - (17 - R)$ $= R + 11$

Because there are 17 red cards in total, the remaining $17 - R$ cards must be in the large pile. All the other cards in the large pile are blue.

Because there are 28 cards in total in the large pile, the number of blue cards in it must be: $28 - (17 - R) = 28 - 17 + R = 11 + R$.

The difference of R red cards in the small pile and $R + 11$ blue cards in the large pile is 11, irrespective of the value of R.

Thinking Further: Answer this question again by first writing a formula for the number of blue cards in the small pile and then a formula for the number of blue cards in the large pile. Do you again see the value $R + 11$?

3. In the language of piles and holes, the opposite of the level state—no piles and no holes—is no holes and no piles. This is again the level state. So, in the context of this model, we do have $-0 = 0$.

4. In the context of whole numbers, a number n is said to be even if $n = 2 \times a$ for some whole number a and odd if $n = 2 \times a + 1$. Because $0 = 2 \times 0$, it fits this definition of being even. Because $-6 = 2 \times (-3)$, it too is even, and -17 is odd because $-17 = 2 \times (-9) + 1$.

VISUALIZING RATIO WORD PROBLEMS

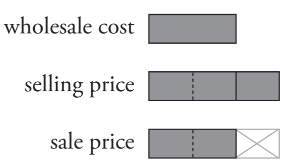
LECTURE 3

The idea of word problems strikes fear in the hearts of many. It is important for a curriculum to have students think through and work through problems described in words, but it is not necessary, or helpful, for word problems to invoke difficult emotional reactions. Instead, make word problems into picture problems. Drawing a diagram for a problem described in words often makes a first step for solving it apparent. And once you have made one piece of progress, the path for finishing the problem often falls into place. In this lecture, you will learn a visual technique for handling tough word problems.

EXAMPLE 1

A furniture store buys couches from a wholesaler and marks up the price of each couch 50% for sale in the store. One weekend the store has a sale: All couches ⅓ off. How much profit, as a percentage, does the store make on each couch during the sale?

3.1



Let's draw a picture. The first strip in **figure 3.1** represents the wholesale cost for a couch. We don't know the number, but the strip can represent the amount.

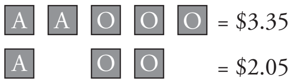
We mark up the price by 50%—that is, we add 50% of our cost to the price.

During the sale's weekend, the price is reduced by ⅓—and we're back to our wholesale purchase price. The store makes no profit on couches during that sale's weekend.

EXAMPLE 2

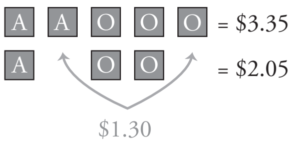
Together, 2 apples and 3 oranges cost \$3.35. Together, 1 apple and 2 oranges cost \$2.05. What is the price of 1 apple?

3.2



We can draw one section of a strip for the cost of each apple and a different strip section for the cost of each orange. See **figure 3.2**.

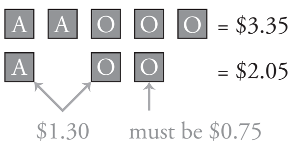
3.3



The top line of the picture has more items in it than the bottom line. Actually, the top line has 1 extra apple and 1 extra orange. And the top collection costs \$3.35 as opposed to \$2.05. That's a difference of \$1.30.

So, we've just figured out that an apple and an orange together cost \$1.30.

3.4



If an apple and an orange cost \$1.30, and the bottom line says that an apple and orange and another orange cost \$2.05, then the extra cost of \$0.75 is coming from the extra orange.

We've just figured out that an orange costs \$0.75.

But the question wants the cost of an apple. We said that an apple and an orange together cost \$1.30, and an orange costs \$0.75. This means that an apple alone costs \$0.55.

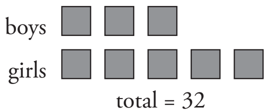
EXAMPLE 3

There are 32 students in a class. The ratio of boys to girls in that class is 3 to 5. How many more girls are there than boys?

Let's first be clear what ratio means: 2 quantities are in an *a*-to-*b* ratio if whenever *a* groups of the first quantity appear in a situation, *b* groups of the second quantity also appear.

In this example, we are told that in a class of 32 two students, there are 3 groups of boys and 5 groups of girls. But we are not told what the group size is.

3.5



So, let's draw a picture. Let's draw 1 block for each group of students: 3 blocks of boys and 5 blocks of girls.

We see now that there are a total of 8 groups of students. With 32 students in all, this means that each group consists of 4 students.

3.6

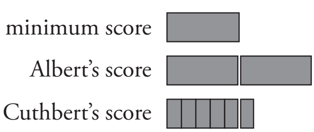
Thus, there are 3 × 4 = 12 boys and 5 × 4 = 20 girls. There are 8 more girls than boys. (Actually, we see directly in the picture that there are 2 more groups of girls than boys—that, again, is 8 girls.)

EXAMPLE 4

Albert and Cuthbert each took an exam. Albert got double the minimum passing score, and Cuthbert got 20% more than the minimum passing score. What is the ratio of Albert's score to Cuthbert's score?

This question is like the one in example 1: There is some basic quantity in the question—a minimum exam score, in this case—and there are changes to that quantity by some percentage amount.

3.7



Let's draw a strip for the minimum exam score. Albert got double it, and Cuthbert got 20% more than it—that is, ⅕ more than it. **Figure 3.7** captures this information. The question wants the ratio of Albert's score to Cuthbert's score.

Albert got points matching double the minimum, and Cuthbert got a score that adds on ⅕ of the minimum.

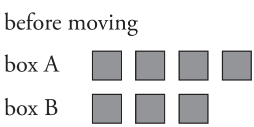
It seems natural to divide Albert's blocks into fifths as well.

We see that Albert's score is worth 10 blocks of points and Cuthbert's score is worth 6 blocks of points. The ratio of Albert's score to Cuthbert's score is 10 to 6. That's the answer to the question! You can call this a ratio of 5 to 3, if you prefer. If you cut the numbers in half, this results in a ratio of 5 to 3, which is equivalent to the ratio of 10 to 6.

EXAMPLE 5

Box A contains 33.333…% (33 ⅓) more coconuts than box B. If half the coconuts are taken from box A and moved to box B, what will be the ratio of the number of coconuts in box A to the number in box B?

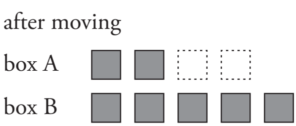
3.8



What we need to do is somehow draw a picture representing box A with 33.333…% more goods in it than box B. So, box B has some goods, and box A has a third more goods. This suggests drawing a diagram showing box B with 3 sections and box A with 1 extra of those sections.

Figure 3.8 is the picture at the start of the question. Then, things were moved: Half the coconuts are taken from box A and moved into box B. **Figure 3.9** the result after moving half the coconuts from box A to box B. We see that the new ratio is 2 to 5.

3.9



EXAMPLE 6

Cuthbert and Filbert each have cash in their wallets. Their cash amounts are in a 3-to-8 ratio, with Filbert possessing more cash than Cuthbert. During a shopping spree, Cuthbert spends ⅓ of his cash and Filbert spends $40. This leaves Filbert with triple the amount of cash in his wallet than Cuthbert. How much money did Cuthbert spend shopping?

3.10

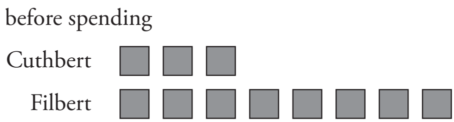
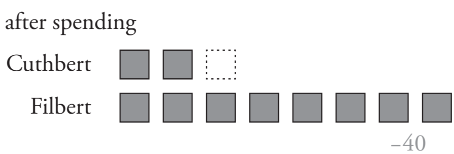


Figure 3.10 is the before-spending picture. Cuthbert has 3 blocks of cash and Filbert has 8.

Next, Cuthbert spends ⅓ of his money and Filbert spends $40. We can see that Cuthbert is left with only 2 blocks of cash. Because we don't know any actual numbers, we don't know how to draw the fact that Filbert spends $40.

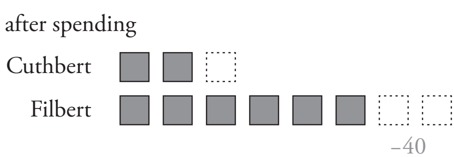
3.11



We'll just write –40 for now. What else can we do?

We are also told that after spending, Filbert has triple the amount of money left than Cuthbert has.

3.12



So, in this after-spending picture, Filbert must have 6 blocks—that's triple Cuthbert. So, taking away $40 must be equivalent to taking away 2 blocks of cash.

How much money did Cuthbert spend shopping? It was 1 block of cash—that's $20.

EXAMPLE 7

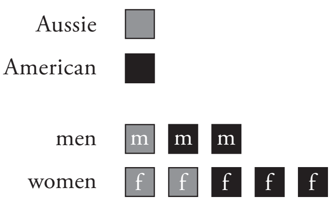
There are 37 people in a room, 14 of whom are Australian and the rest are American.

Of the men in the room, ⅓ are Australian.

Of the women in the room, ⅖ are Australian.

How many Aussie men are there? How many Aussie women are there?

3.13



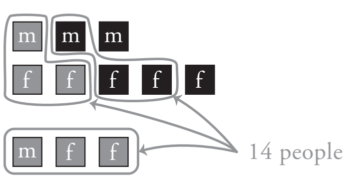
We have 4 types of people: Aussie men, American men, Aussie women, and American women. We know that ⅓ of the men are Aussies, and ⅖ of the women are Australian. Here's a picture encoding all of that. The male blocks are all the same size (so that we can see that the Aussie men are ⅓ of all men), and the female blocks are all the same size (so that we can see that ⅖ of the women are Aussie). But we don't claim that a male block of people is the same size as a female block of people.

3.14

We also know that there are 14 Australians in total.

The second picture says that 14 people equals 1 male block and 2 female blocks. We see 2 copies of this in the first picture.

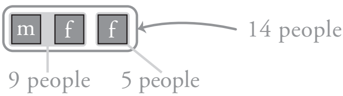
3.15



Actually, this accounts for 28 people. There are 37 people in total. So, the remaining 1 block of American males and 1 block of American females must add to the remaining 9 people.

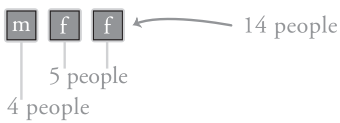
So, we now know that a male block plus a female block equals 9 people. Can we use that? We just focused on the first picture. What if we go back and focus on the second picture? We know that 1 male block and 1 female block makes 9 people. We see now that a female block must consist of 5 women.

3.16



The other block of women is 5 people, too. So, this means that the male block consists of 4 people.

3.17



These are all the Australians—which is what the question asked about. We see that there are 4 Aussie men and 10 Aussie women in that room of 37 people.

FURTHER EXPLORATION

WEB

Tanton, "An Aside on Tape Diagrams."
http://gdaymath.com/lessons/powerarea/1-8-connection-to-tape-diagrams/

PROBLEMS

1. At a party, 2⁄7 of the men present have a first name that begins with a vowel, as do 3⁄8 of the women present, and these 2 counts of people are equal. What fraction of the people at the party have a first name that begins with a vowel?

2. Dirk is currently twice the age that Shivram was when Dirk was Shivram's age. If the sum of the men's 2 ages is currently 63, how old is Dirk today?

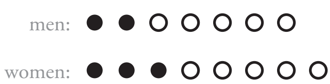
3. If 2 ½ pies are shared equally among 3 ⅓ boys, how much pie does each (full) boy receive?

4. If it takes 6 women 8 days to build 3 houses, how long will it take 8 women to build 9 houses?

SOLUTIONS

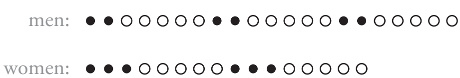
1. In the following schematic, each dot represents a group of people of some fixed size, with the colored dots representing those with a first name that begins with a vowel.

3.18



We are told that the counts of men and women with special first names are equal. To match the counts of shaded dots, triple the number of dots in the picture for the men and double the number of dots in the picture of the women.

3.19



We now see that 6 out of 37 people at the party have a first name that begins with a vowel.

2. Dirk is some number of years older than Shivram. If we subtract that difference from each of the men's ages, then Dirk's current age is double Shivram's age in the past.

3.20

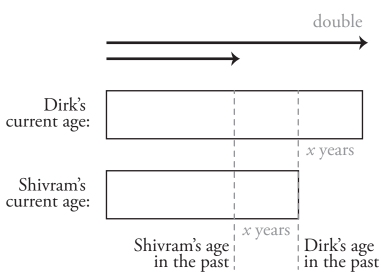


Figure 3.20 captures this information.

If x is the current difference in their ages, then we see from the diagram that Dirk is $4x$ years old and Shivram is $3x$ years old.

The sum of their ages is 63; thus, $4x + 3x = 63$, giving $x = 9$. Dirk is 36 years old, and Shivram is 27 years old.

3. Following the thinking of question 1, doubling the number of pies and the number of boys will not change the proportions of this sharing problem—nor will tripling the number of pies and the number of boys. Do both!

2 ½ pies for 3 ⅓ boys is equivalent (via doubling) to 5 pies for 6 ⅔ boys, which is equivalent (via tripling) to 15 pies for 20 boys.

Each boy gets ¾ of a pie.

4. Let's extend this proportional reasoning further. Follow this schematic:

Women	Days	Houses
6	8	3

It will take those 6 women 3 times as long to build triple the number of houses.

If just 1 woman were doing all the work, it would take 6 times longer again.

But if 8 women are working, we can decrease the amount of time needed by a factor of 8.

Women	Days	Houses
6	8	3
6	24	9
1	144	9
8	18	9

So, 8 women can build 9 houses in 18 days.

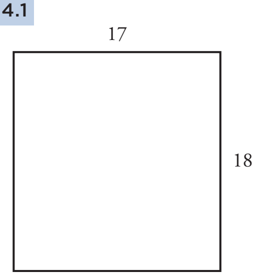
VISUALIZING EXTRAORDINARY WAYS TO MULTIPLY

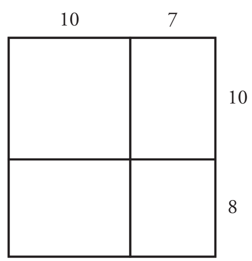
LECTURE 4

The mathematics you have learned about so far has been addition (adding quantities); subtraction (adding negative quantities); and some ratios and proportions, which are forms of multiplication. And you went extraordinarily far with just those ideas, looking at and solving tricky problems. But in this lecture, you will make sense of multiplication by exploring the area of a rectangle: By chopping a rectangle into pieces, you can make the multiplication much easier by adding the area of the individual pieces.

MULTIPLICATION WITH RECTANGLES

4.1





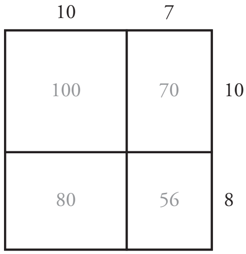
Multiplication is a problem in geometry. When someone asks you to compute 17 × 18, think area. You are being asked to find the area of a rectangle that is 17 units wide and 18 units high.

Those numbers, 17 and 18, are difficult, so let's avoid hard work and split the numbers into more manageable figures. For example, we can think of 17 as 10 + 7 and 18 as 10 + 8. This means that we're thinking of dividing the rectangles into 4 pieces.

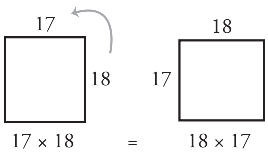
And it is easy for us to work out the areas of the individual pieces: 10 × 10 gives a piece of area 100, 7 × 10 gives a piece of area 70, 10 × 8 gives a piece of area 80, and 7 × 8 gives a piece of area 56.

So, the area of the rectangle, 17 × 18, must equal 100 + 70 + 80 + 56 = 100 + 150 + 56 = 306.

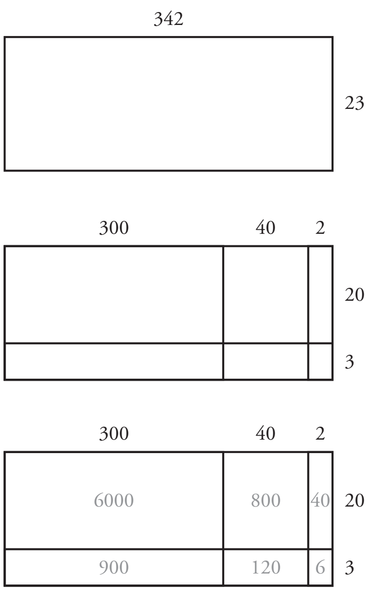
4.2



4.3



4.4



People like to believe that multiplication is commutative—that is, that the order in which one computes a product does not matter: 18 × 17, apparently, gives the same answer as 17 × 18. The visual makes it clear why this is so. Turn the 17 × 18 rectangle 90° and suddenly we see an 18 × 17 rectangle. The area of the rectangle hasn't changed, so the answers to the 2 multiplication problems just have to be the same.

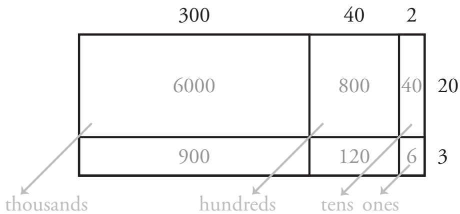
Consider another example: 342 × 23. Again, it's an area problem. (See **figure 4.4**.)

Let's break the rectangle into manageable pieces: 300 and 40 and 2, and 20 and 3.

We can work out the areas of the individual pieces of the rectangle quite easily. We can see 6000, 800, 40, 900, 120, and 6. The area of the rectangle must be the sum of these values.

Before we compute this sum, let's point out that there is some lovely structure in this picture: The pieces of the rectangle line up diagonally by thousands, hundreds, tens, and ones. (See **figure 4.5**.)

4.5



$342 \times 23 = 6000 + 1700 + 160 + 6 = 7866$

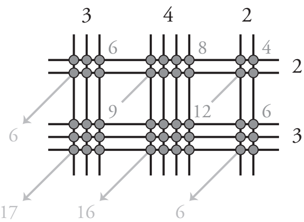
We have 6000 plus 800 and 900, making 1700; 40 + 120 making 160, which is 16 tens; and 6.

Thus, $342 \times 23 = 6000 + 1700 + 160 + 6 = 7866$.

Let's compare this to the line method.

We draw 3 lines, 4 lines, and 2 lines for 342, and 2 lines and 3 lines for 23, and then count intersection points. We add those counts diagonally and read off the answer in terms of thousands, hundreds, tens, and ones.

4.6



$342 \times 23 =$
6 thousands, 17 hundreds, 16 tens, and 6
$= 7866$

In the rectangle, we compute 300×20 to get 6000 in the top-left corner. In the line method, we compute 3×2 to get 6 intersection points in the top-left corner.

In the rectangle, in the bottom-left corner we compute 300×3 to get 900. In the line method, we compute 3×3 to get 9 intersection points in the bottom-left corner.

The computations are identical throughout. The rectangle method keeps tracks of the thousands, hundreds, tens, and ones as you go along. The line method ignores the thousands, hundreds, tens, and ones at first, but then it adds the results diagonally, now keeping the thousands, hundreds, tens, and ones in mind, just as one should.

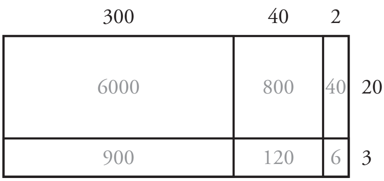
The line method is just encoding all the information of the rectangle, so really is doing correct multiplication.

In fact, the standard algorithm, long multiplication, is just the rectangle in disguise!

4.7

Sometimes long multiplication is taught in a less compact form.

$$\begin{array}{r} 342 \\ \times\ 23 \\ \hline 6 \\ 120 \\ 900 \\ 40 \\ 800 \\ 6000 \\ \hline 7866 \end{array}$$



For example, for 342 × 23, we write 2 × 3 = 6. Then, we look at 4 × 3 but note that it is really 40 × 3, so we write the answer, 120. Then, we look at 3 × 3 but note that it is really 300 × 3 and write 900—and so on.

This version of long multiplication is the same as the rectangle, listing all the pieces and adding them. It just doesn't draw the rectangle.

The compact version of the long multiplication is designed to save space. It writes all the intermediate computations on top of one another.

3 × 2 is 6, which we write.

3 × 4 is really 3 × 40, which equals 120. That's 2 tens, which we write, and 100, which we'll delay writing until we work with the hundreds.

4.8

$$\begin{array}{r} {}^{1}342 \\ \times\ 23 \\ \hline 1026 \end{array} \qquad \begin{array}{r} 342 \\ \times\ 23 \\ \hline \boxed{\begin{array}{r} 6 \\ 120 \\ 900 \end{array}} \\ 40 \\ 800 \\ 6000 \\ \hline 7866 \end{array}$$

We finish off the first row by doing 3 × 3 but note that this is really 3 × 300, so the answer is 900, but we still have the 100 we held off on writing down. So, we really have 10 hundreds. Let's write 10 in that same row, for 10 hundreds.

4.9

$$\begin{array}{r} {}^{1}342 \\ \times\ 23 \\ \hline 1026 \\ 6840 \\ \hline 7866 \end{array} \qquad \begin{array}{r} 342 \\ \times\ 23 \\ \hline 6 \\ 120 \\ 900 \\ \boxed{\begin{array}{r} 40 \\ 800 \\ 6000 \end{array}} \\ \hline 7866 \end{array}$$

Next, we do the same thing again: 2 × 2 is really 20 × 2. So, let's put in a 0 to capture the idea that we're now working with tens. 20 × 2 is 40, that's 4 tens, so we write 4 in the tens place.

20 × 40 is 800, so we write 8 in the hundreds place. 20 × 300 is 6000, so we write 6 in the thousands place.

Now add all the pieces and get 7866, again, as the final answer. It is so much easier to just do the rectangle!

USING THE RECTANGLE METHOD

The rectangle method is rich with understanding. We can use it to explain an age-old question: Why, in mathematics, is negative times negative computed as positive?

To get there, let's check all of the other possibilities first.

We're okay with positive times positive. For example, 2 × 3 is usually interpreted as 2 groups of 3 (3 + 3), and that makes 6—positive 6.

We're okay with positive times negative: 2 × (–3) is 2 groups of –3 (–3 + –3), which would be –6.

But things are a bit shaky with negative times positive: for example, (–2) × 3. The idea of "negative 2 groups of 3" doesn't really make sense.

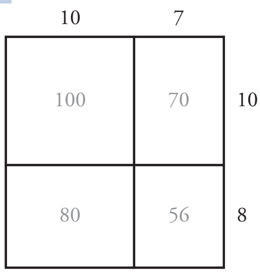
But there is a way out of this pickle. We like to believe that multiplication is commutative—that is, that you can switch order in a multiplication problem and get the same answer in the end. And most people think that this should be true of all numbers, including negative ones. So, we can switch the order of the –2 × 3 and think of it as 3 × –2. And that is something we can interpret: 3 groups of –2 (–2 + –2 + –2) makes –6.

So, positive times positive is positive; positive times negative is negative; and negative times positive is negative. Now comes the big question: What is negative times negative? What is -2×-3, for example?

Switching the order doesn't help: -3×-2 is just as confusing as -2×-3. So, we need something more.

Most people like to believe that all of the usual rules of arithmetic should hold for all types of numbers, even negative ones.

4.10



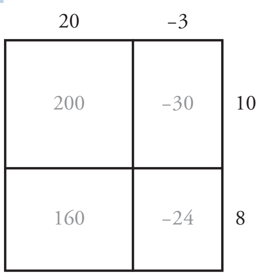
$17 \times 18 = 100 + 70 + 80 + 56 = 306$

This means that the arithmetic that is encoded in our area model should hold for negative numbers, as well. Even though geometry doesn't allow objects to have negative lengths, the arithmetic that pictures with negative side lengths represent should still be valid.

Here's our picture for 17×18. The answer is 306.

But let's be quirky and change how we think of 17. Instead of thinking of it as $10 + 7$, think of it as $20 + -3$.

4.11



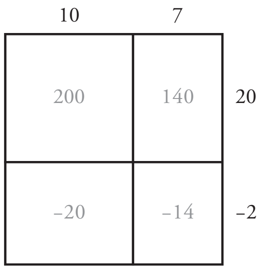
$17 \times 18 = 200 + 160 - 30 - 24 = 306$

As a piece of geometry, this is strange. But look at the arithmetic. We have $20 \times 10 = 200$.

We have $20 \times 8 = 160$. We have -3×10, which we can do; negative times positive is negative, so the answer is -30. And we have -3×8, giving -24.

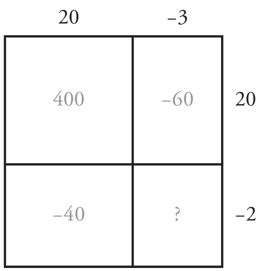
We get $200 + 160 - 30 - 24$, and that is 306 again, just as it should be. The strange geometry picture is still representing correct arithmetic.

4.12



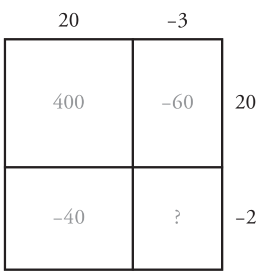
$17 \times 18 = 200 + 140 - 20 - 14 = 306$

4.13



$17 \times 18 = 400 - 60 - 40 + ? = 306$

4.14



$17 \times 18 = 400 - 60 - 40 + \boxed{?} = 306$

must be 6

Instead, we can be quirky with the number 18 and think of it as 20 + −2.

We still get 306.

$10 \times 20 = 200$; $7 \times 20 = 140$;
$10 \times -2 = -20$; $7 \times -2 = 14$.

These pieces again add to 306.

Let's be really quirky and write 17 as 20 − 3 and 18 as 20 − 2 simultaneously.

What do we get?

$20 \times 20 = 400$; $20 \times -2 = -40$;
$-3 \times 20 = -60$.

Now we see that we have a problem: We need to work out -2×-3 or -3×-2, negative times negative.

Just prior to this we worked out 17×18 in 3 different ways and got the answer 306 every time. If we want mathematics to be consistent, then we should get the answer 306 in this fourth example, too. In this case, we see that we have no choice but to declare -2×-3 to be positive 6.

So, if we choose to believe that all numbers, including negative ones, obey the standard rules of arithmetic, then for consistent mathematics, we have no choice but to set negative times negative to be positive.

FURTHER EXPLORATION

WEB

Tanton, "A Cute Finger (and Toe!) Multiplication Trick."
http://www.jamestanton.com/?p=542
———, "Line Multiplication."
http://www.jamestanton.com/?p=1122
———, "Multiplication without Multiplying."
(Yet another mysterious multiplication method.)
http://www.jamestanton.com/?p=1172
———, "Why Is Negative Times Negative Positive?"
http://www.jamestanton.com/?p=353

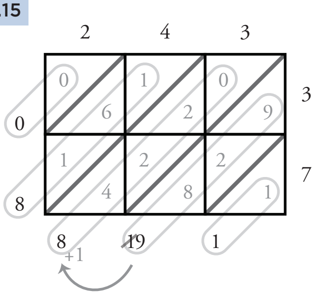
READING

Tanton, *Mathematics Galore!*
———, *Thinking Mathematics! Vol. 1.*

PROBLEMS

1. In some curricula, students are taught the lattice method for performing long multiplication.

4.15



243 × 37 = 8991

To multiply 243 and 37, for example, draw a 2-by-3 grid of squares and write the digits of the first number at the head of each column and the digits of the second number at the end of each row. Divide each cell of the grid with a diagonal line and write the product of the column digit and row digit of that cell as a 2-digit answer that is placed on either side of that diagonal. Add the digits in each diagonal and conduct any necessary carries.

a Compute 251 × 133 via the lattice method.

b Why does this lattice method work?

2. Neptunians have 2 hands, which each have 4 fingers. To compute products up to 8 times 8, they use the following finger trick:

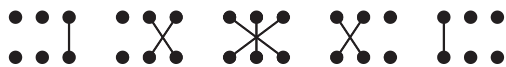
> A closed fist represents 4. To represent a number between 4 and 8, raise 1 finger for each count to bring the number 4 up to the desired value. (Thus, 6 is represented as 4 + 2, which is 2 fingers raised on a closed fist, and 7 is represented as 4 + 3, which is 3 fingers raised on a closed fist.)
>
> To compute a product of 2 numbers, represent each number on a hand. Count the total number of raised digits, give each the value 8, and compute that value of that many 8s. Multiply the unraised digits on each hand and add this product to the count of 8s. This value is the answer to the original multiplication problem.

For example, in computing 6 × 7, there are 5 raised digits in total, making 5 8s, or 40. There are 2 and 1 unraised digits in each hand, respectively. Adding 2 × 1 = 2 to 40 gives 42. This is indeed the answer to 6 × 7.

Why does this method work?

3. Vedic mathematics, established in 1911 by Jagadguru Swami Sri Bharati Krishna Tirthaji Maharaja, has students compute the product of two 3-digit numbers as follows:

4.16

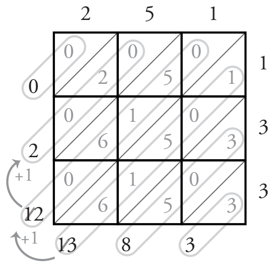


What do you think this sequence of diagrams means?

SOLUTIONS

1. **a** Use a 3-by-3 table this time.

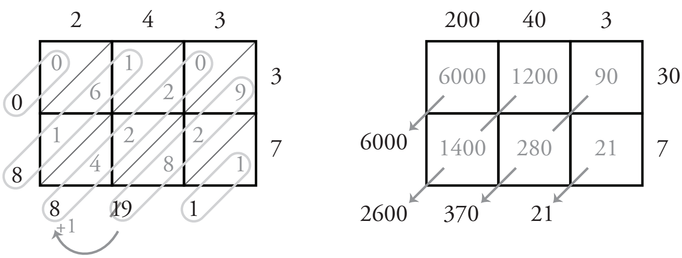
4.17



$251 \times 133 = 33{,}383$

b The lattice method is the area method in disguise. Look at the example given in the question computed via both methods side by side as in **figure 4.18**.

4.18



$243 \times 37 = 8991$

We see that the diagonal lines serve to separate tens from ones, hundreds from tens, thousands from hundreds, and so on, within each cell and therefore organize powers of 10 in a diagonal fashion. This accounting system allows you to let go of drawing all the zeros for the powers of 10 (whereas the area model keeps track of the powers of 10 by writing all the zeros).

2. Suppose that a digits are raised on 1 hand (to represent the number $4 + a$) and b on the other (to represent the number $4 + b$). Then, there are $a + b$ raised digits in all, with $4 - a$ unraised digits on the first hand and $4 - b$ unraised on the second.

We claim that the product $(4 + a)(4 + b)$ can be computed as $8(a + b) + (4 - a)(4 - b)$. Let's check.

$$(4 + a)(4 + b) = 16 + 4a + 4b + ab$$
$$8(a + b) + (4 - a)(4 - b) = 8a + 8b + 16 - 4a - 4b + ab$$
$$= 4a + 4b + 16 + ab$$

These are indeed the same!

With n fingers on each hand, it is true that $(n + a)(n + b) = 2n(a + b) + (n - a)(n - b)$.

Thinking Further: Mercurians have 2 hands, 1 with 5 fingers and 1 with 7 fingers. Is there a (nonsymmetrical) multiplication trick they can use?

3. This is a visual mnemonic for long multiplication. The first diagram says to multiply the units; the second diagram says to multiply units and tens to get answers in tens; the third diagram says to compute all the products that give hundreds; the fourth diagram says to compute all the products that give thousands; and the fifth diagram says to compute all the products that give ten thousands.

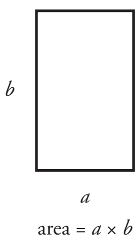
VISUALIZING AREA FORMULAS

LECTURE 5

Even seemingly simple ideas can serve as portals to surprisingly sophisticated insights. In this lecture, you will explore the topic of area and dive into it deeply. You will examine what you know about area from your school days—but with a twist so that you can see the topic in a new, richer light. The rectangle is the starting point of this lecture. Chopping this figure into pieces and looking at the areas of the parts is a powerful idea that leads to some deep mathematics.

CALCULATING AREA

5.1

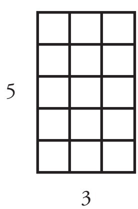


If a rectangle has base a and height b, then its area is $a \times b$.

This makes good intuitive sense. If a rectangle is 3 units wide and 5 units high, then we see that 3 × 5, or 15, unit squares fit inside the figure. Its area is indeed 15 square units.

This idea feels right even if we don't work with whole numbers. Working with fractional side lengths will involve fractional parts of unit squares. But that's okay. The area of the rectangle is still base × height.

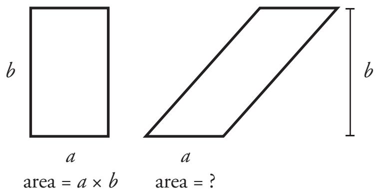
5.2



Suppose that we take our rectangle and push it slightly to the side, creating a tilted figure. This new figure has pairs of parallel sides and is called a parallelogram. But has the area changed? (See **figure 5.3**.)

The area of a parallelogram is base × height—that is, its area follows the same formula as that of a rectangle. Can this really be true? Is the area of a parallelogram the same as the area of a

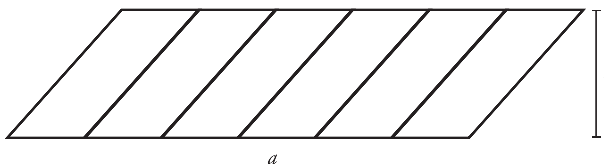
5.3



rectangle? If we look at the picture, it looks like the sideways shearing has created some additional area.

Let's compare the areas by taking lots of copies of the parallelogram, with base a and height b, and line them up in a row.

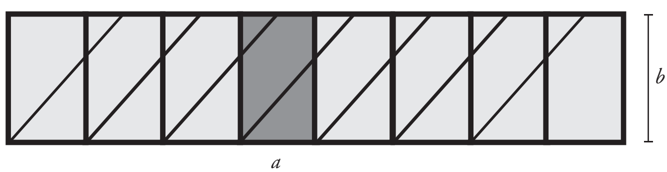
5.4



Then, draw a copy of the rectangle, with base a and height b.

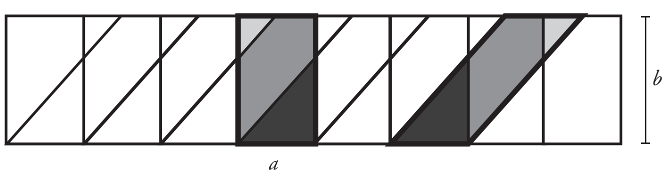
In fact, draw in lots of copies of this rectangle!

5.5



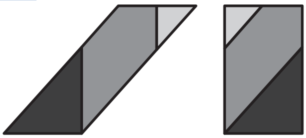
Now let's shade some matching pieces.

5.6



We see now that the rectangle and the parallelogram are composed of identical matching pieces, which means that they do have the same area.

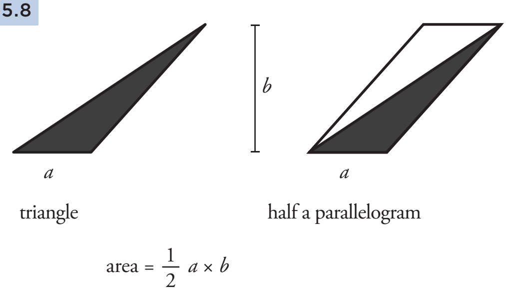
5.7



Actually, this picture does more than just tell us that the areas are the same; in fact, it gives us instructions on how to take a paper copy of a parallelogram, cut it into pieces, and rearrange those pieces to make a rectangle.

We are taught in school that the area of a triangle is half its base times its height—that is, the area of a triangle is given by half the formula for the area of a parallelogram. And we can see that this is true: A triangle is half a parallelogram.

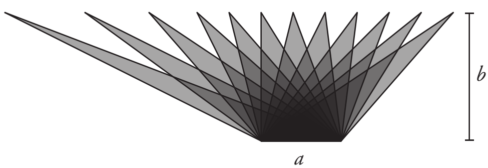
5.8



Its area must be ½ base × height.

The formula says that all triangles with the same base a and the same height b must have the same area—½ $a \times b$. This seems difficult to believe, but it must be true.

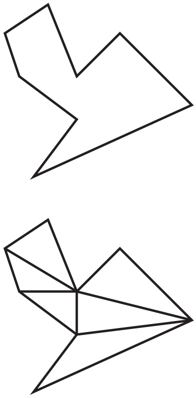
5.9



This means that we can slide the top of a triangle left and right, and as long as we don't change the height of the triangle, the area of that triangle never changes. All the triangles in this picture have the same area.

THE AREA OF A POLYGON

5.10



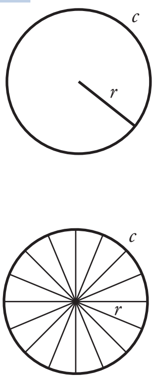
The area of a rectangle is base × height. The area of a parallelogram is base × height. Triangles are half parallelograms and therefore have area ½ base × height.

Because every polygon can be subdivided into triangles, and we have a formula for computing the area of each triangle, we now have the means to compute the area of any polygon.

All the area formulas for special polygons that you might have memorized in school are just applications of this single idea. Once you have the area of a triangle in your mind, you really have all the area formulas in your mind.

THE AREA OF A CIRCLE

5.11



There is one additional area formula: the formula for the area of a circle. But a circle is not a polygon. So, how do we think about the area of a circle? There are no obvious triangles sitting in circles.

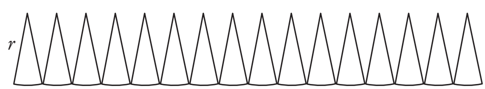
Consider a circle of radius r. Let's call the length of its circumference C.

We can slice the circle into triangular pieces.

The pieces aren't really triangles—they have curved bases. But if we do lots of thin slices, the pieces can be approximated as triangles.

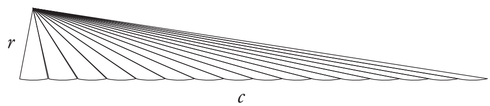
Next, pull apart our almost-triangular slices and line them up in a row.

5.12



Now slide the tops of each of these triangular pieces to the left. As long as we keep the heights the same, the areas won't change.

5.13



Now we see just one big triangle with height, the radius, r and base matching the circumference of a circle, C. The base is bumpy, so this is only an approximation, but if we were to do thinner and thinner slices, this picture would become more and more closely the picture of a triangle of height r and base C.

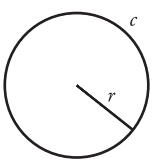
The area of this triangle is ½ base × height—that is, $\frac{1}{2}\, r \times C$.

But these slices make the circle. So, the area of the circle must be $\frac{1}{2} \times r \times C$. This is the area formula for a circle. All the approximation pictures approach this formula.

But this probably isn't the area formula you remember from school days. You probably have πr^2 in your mind.

To get this version of the formula, people use the fact that circumference C equals $2 \times \pi \times r$. (This is actually just the definition of π rearranged.) Putting this into our formula gives the area of a circle as $\frac{1}{2} \times r \times 2 \times \pi \times r$. Rearranging and simplifying, this is πr^2.

5.14



$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{C}{2r} \longrightarrow C = 2\pi r$$

$$\text{area} = \frac{1}{2} \times r \times C = \frac{1}{2} \times r \times 2\pi r = \pi r^2$$

All of this comes from rectangles—from dividing rectangles into pieces, rearranging those pieces into parallelograms, and recognizing triangles as half parallelograms. And once you have formulas for the areas of triangles, you have it all, even the areas of circles if you are willing to do better and better approximations.

THE BOLYAI-WALLACE-GERWIEN THEOREM

This idea of dividing a figure into pieces and rearranging those pieces to get a new shape of the same area is actually mathematically very deep. For example, in the 1790s Hungarian mathematician Farkas Bolyai asked the reverse question: Given 2 polygons of the same area, is it always possible to divide one of the polygons into pieces and rearrange those pieces to form the second polygon?

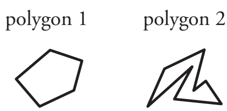
He couldn't answer the question. It was a difficult question, despite the fact that is seems that it should obviously be true.

After almost 2 decades, Scottish mathematician William Wallace proved that for 2 polygons of the same area it is always possible to dissect one into pieces that can be rearranged to form the other. Unware of Wallace's proof, Paul Gerwien proved the result again 25 years later.

In the late 1800s, German mathematician David Hilbert asked the analogous question for 3-dimensional shapes: If given 2 polyhedra of the same volume, is it possible to dissect one polyhedron into pieces, rearrange those pieces, and form the second polyhedron? In 1900, Max Dehn proved that the answer, shockingly, is no.

So, there is something special about 2-dimensional figures: that a limit on dimension allows us to take paper and scissors to transform polygons of the same area into each other, but with the extra freedom of the third dimension, this is no longer true.

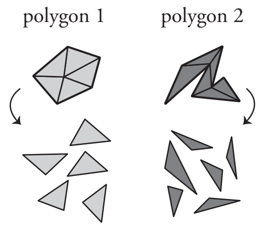
5.15



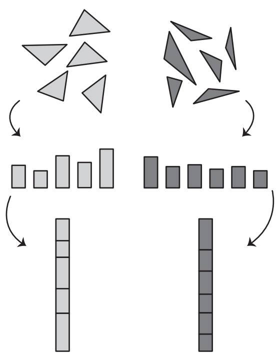
Let's prove the 2-dimensional case.

Imagine that we have 2 different polygons of the same area: polygon 1 and polygon 2.

5.16



5.17



We can certainly cut each polygon into a collection of triangles.

Let's figure out a way to cut each triangle into pieces that turn each triangle into a rectangle of a certain fixed width. If we do that for each triangle in polygon 1, changing each to a rectangle of a certain base, and do the same for each triangle in polygon 2, changing each to a rectangle of that same base, then we can stack the rectangles on top of each other to turn polygon 1 into a big rectangle and to turn polygon 2 into a big rectangle.

Because the 2 original polygons have the same area, these 2 big rectangles have the same area, too. And because their widths are the same, the 2 rectangles have the same height and therefore are identical.

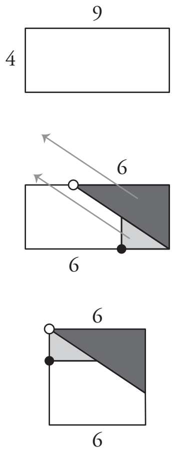
And that means that we're done! To convert polygon 1 into polygon 2, chop polygon 1 into triangles. Chop each triangle into pieces and convert it into a rectangle of some fixed width. Stack the rectangles to turn polygon 1 into one big rectangle.

Now do on that rectangle what would be the reverse process for polygon 2. Turn the big rectangle back into the triangles that are right for polygon 2, and then make polygon 2 with those triangles.

This will create a lot of tiny pieces, but all the pieces we create come from polygon 1 and can be used to create polygon 2. This proves the Bolyai-Wallace-Gerwien theorem—except for the one key component. It all rests on our ability to convert a triangle into a rectangle of given width.

Can that always be done? The answer is yes.

5.18

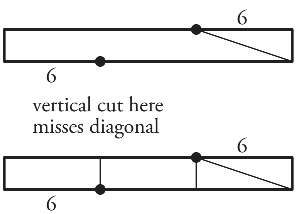


As a warm-up for finishing off the proof of the Bolyai-Wallace-Gerwien theorem, let's learn how to turn any rectangle into a square.

Here's a rectangle that is 9 inches long and 4 inches high. It has an area of 36 square inches. If we convert it into a square, the square will have an area of 36 square inches, too, and therefore will have a side length of 6 inches.

This method works for almost all rectangles. We lucked out that our vertical cut hit the diagonal line. If the rectangle is very skinny, such as a 2-inch by 18-inch one, still of area 36 square inches, we will have to chop off several 6-inch lengths first to then perform the construction.

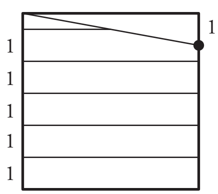
5.19



We know how to turn rectangles into squares. But can we turn squares into rectangles? That's a silly question—squares are already rectangles. Is it always possible to convert a square into a rectangle with a prespecified base?

For example, **figure 5.20** shows a 5.5-by-5.5 square. Let's convert it into a rectangle of width 1 inch.

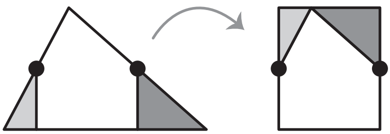
5.20



So, we've proved that it is possible to take any rectangle and convert it into a rectangle that is 1 unit wide: First turn the rectangle into a square and then convert the square into a rectangle that is 1 inch wide.

So, we now know that all rectangles can be converted into new rectangles that are just 1 unit wide.

5.21



Let's go to the final piece of the Bolyai-Wallace-Gerwien theorem, to prove that it is always possible to convert a triangle into a rectangle of a given width.

Every triangle can be cut into pieces that rearrange into a rectangle.

If this new rectangle happens to be the width we want, great! If it isn't, then we convert the rectangle into a square and then convert the square into a rectangle of the width we want.

So, given 2 polygons of equal area, it is always possible to chop the first polygon into pieces that can be rearranged into the second: Chop it into triangles, rearrange each triangle into a rectangle that is 1 unit wide, and stack those rectangles to make one big rectangle. Then, chop that big rectangle into pieces that reverse the process for polygon 2.

FURTHER EXPLORATION

WEB

Bogomolny, "Wallace-Bolyai-Gerwien Theorem."
http://www.cut-the-knot.org/do_you_know/Bolyai.shtml

Tanton, "Cool Math Essay: February 2016."
(Feynman's triangle problem—and beyond.)
http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_February-2016_Inner-Triangles.pdf

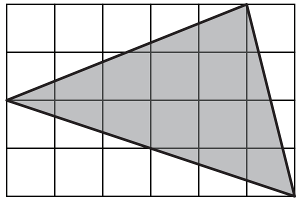
READING

Frederickson, *Dissections.*

Tanton, et al, "Pick's Theorem—and Beyond!"

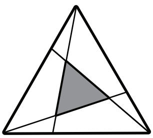
PROBLEMS

5.22



1. What is the area of the shaded triangle shown in **figure 5.22**? (Each square in the grid has an area of 1 square unit.)

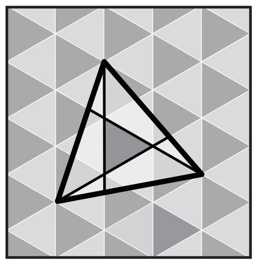
5.23



2. Lines are drawn from each corner of an equilateral triangle to a point ⅓ of the way along the opposite side, as shown in **figure 5.23**. Compared to the area of the whole triangle, what is the area of the shaded triangle that is formed by these lines?

   Hint: Imagine this diagram as drawn on a kitchen floor tiled with triangular tiles. Align matters so that the inner shaded triangle matches a kitchen tile.

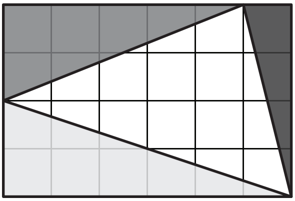
5.24



SOLUTIONS

1. It is (much!) easier to compute the area outside of the triangle within the rectangle.

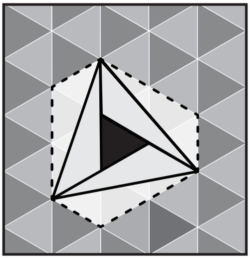
5.25



The rectangle has area 4 × 6 = 24 square units. The region within the rectangle outside of the triangle is composed of 3 right triangles of areas ½ × 2 × 5 = 5, ½ × 1 × 4 = 2, and ½ × 2 × 6 = 6 square units. Thus, the area of the region we seek is 24 – 5 – 2 – 6 = 11 square units.

2. The central shaded triangle is surrounded by 3 triangles, and each of those triangles is half a parallelogram.

5.26



Each parallelogram is composed of 4 kitchen tiles and therefore has an area of 4 kitchen tiles. Thus, the area of the original large equilateral triangle matches 2 + 2 + 2 + 1 = 7 kitchen tiles.

The inner shaded triangle has an area that is 1⁄7 the area of the original triangle.

Note: There is no need for the original triangle to be equilateral.

THE POWER OF PLACE VALUE

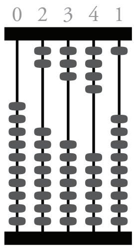
LECTURE 6

An abacus is an ancient counting device that is still used by merchants in some parts of the world. We can use the beads on rods to record numbers, and by sliding the beads along the rods, we can change the numbers and thereby do computations. In this lecture, the abacus will be used as an inspiration for all the arithmetic and computation we are going to be doing for the rest of the course.

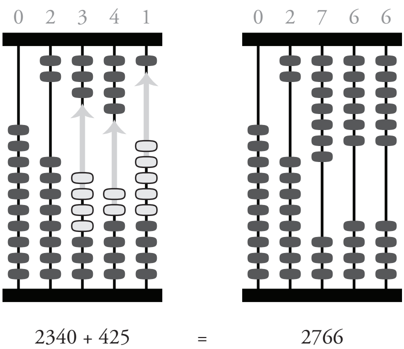
USING AN ABACUS

An abacus has 10 beads on each of a series of rods. Each rod represents a place value for a number, and the beads on that rod are used to represent the digit in that place value—from 0 to 9. For example, **figure 6.1** shows the number 2341 on a simple abacus. We see 2 thousands, 3 hundreds, 4 tens, and 1 one.

6.1



6.2

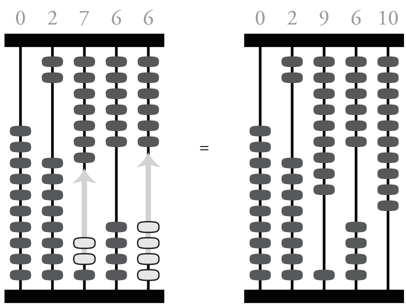


If we want to add 425 to this the number, for example, just slide up 4 more beads in the hundreds place, 2 in the tens place, and 5 in the ones place. We see the answer 2766 appear. (See **figure 6.2**.)

Now suppose that we want to add the number 204 to this answer of 2766. We can slide 2 beads up in the hundreds place and 4 up in the ones place. But we get an answer that doesn't sound right: 2 thousands, 9 hundreds, 6 tens, and 10 ones.

But we realize that 10 ones are the same as 1 ten. So, we slide the rightmost set of 10 beads down and slide up 1 bead in the tens place in their stead: 10 ones are replaced by 1 ten.

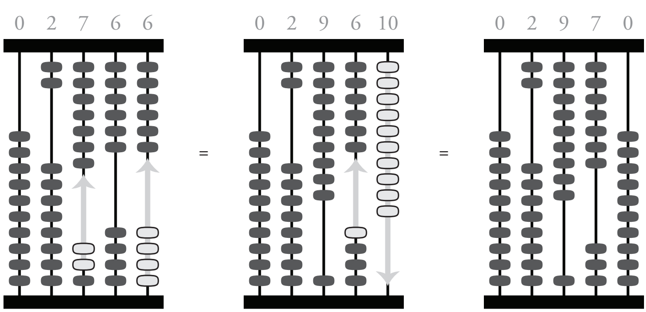
6.3



2766 + 204 = ?

We now see the answer, 2970.

6.4

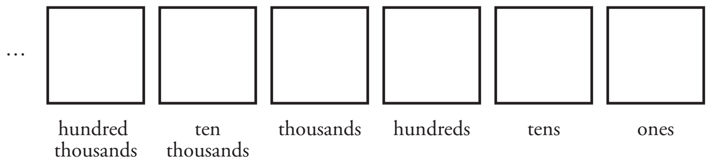


2766 + 204 = ? = 2970

DOTS AND BOXES

We can use the abacus as an inspiration for a visual idea to explain all arithmetic. Instead of rods, let's draw boxes. Just as rods in an abacus represent a place value—ones, tens, hundreds, thousands, and so on—each of the boxes represents a place value, too.

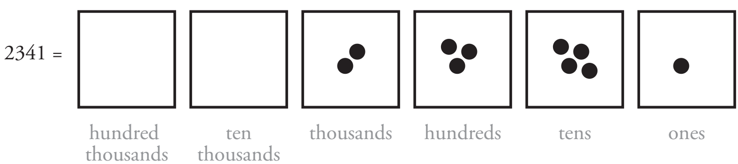
6.5



We can have as many boxes as far to the left as we please—that is, we can have as many place values as far as we are willing to go.

Instead of drawing beads, we can draw dots, which go into the boxes and give the digits of the numbers we wish to represent. For example, here is the number 2341 with dots and boxes.

6.6

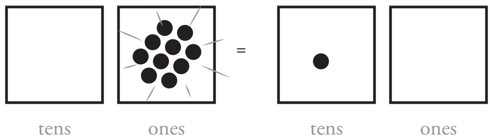


This schema of dots and boxes is dynamic. If we ever find 10 dots in a box, they explode—and disappear—to be replaced with 1 dot 1 place to the left.

For example, if we ever see 10 dots in the units place, they explode and become 1 dot in the tens place: 10 ones is the same as 1 ten.

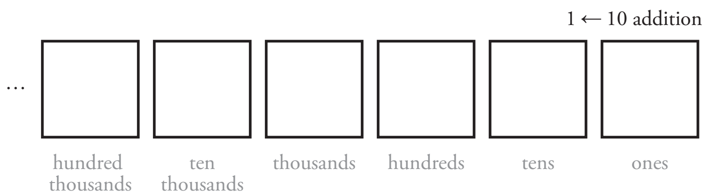
In general, 10 dots in any box always explode and disappear to become 1 dot that is 1 place to the left.

6.7



To express this dynamic action, we can call this dots-and-boxes schema a machine, a 1 ← 10 machine, written with the backward arrow to indicate how dots explode.

6.8



1 ← 10 ADDITION

Let’s go through the algorithms that you were taught in school but use the 1 ← 10 machine to make sense of them.

6.9

$$\begin{array}{r} 273 \\ \underline{+512} \end{array}$$

Consider the sum 273 + 512. The long-addition algorithm you were probably taught had you write matters this way: Line up the numbers by columns.

You were taught to compute this sum from right to left, which seems odd, because we read from left to right.

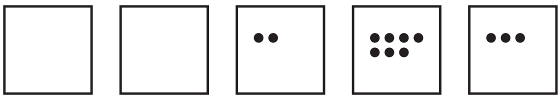
6.10

$$\begin{array}{r} 273 \\ +512 \\ \hline 7 \end{array} \qquad \begin{array}{r} 273 \\ +512 \\ \hline 78 \end{array} \qquad \begin{array}{r} 273 \\ +512 \\ \hline 785 \end{array}$$

If we do the summations from left to right instead—2 + 5 = 7, 7 + 1 = 8, and 3 + 2 = 5—we get the answer 785, which is correct.

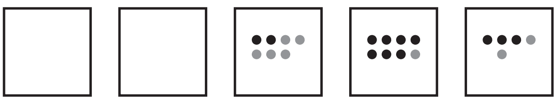
And philosophically, this approach is correct. In a 1 ← 10 machine, 273 looks like this: 2 dots, 7 dots, 3 dots.

6.11



And if we add 512 to it—that is, 5 more hundreds, 1 more ten, and 2 more ones—we get 7 hundreds, 8 tens, and 5 ones.

6.12



So, going left to right was just fine.

6.13

$$\begin{array}{r} 258 \\ +167 \\ \hline \end{array}$$

Of course, additions can be more complicated than this example.

Let's try 258 + 167.

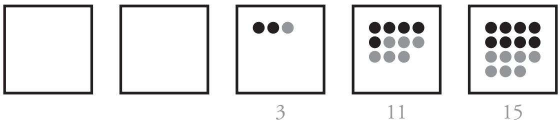
6.14

$$\begin{array}{r} 258 \\ +167 \\ \hline 3 \end{array} \qquad \begin{array}{r} 258 \\ +167 \\ \hline 3^{11} \end{array} \qquad \begin{array}{r} 258 \\ +167 \\ \hline 3^{11\,15} \end{array}$$

Going from left to right, we get: 2 + 1 = 3, 5 + 6 = 11, and 8 + 7 = 15. The answer is three hundred, eleven-ty, fifteen.

That answer is mathematically correct and valid. The dots and boxes show that it is so.

6.15

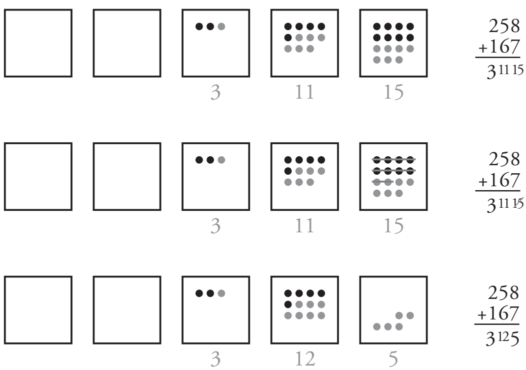


If we start with 2 hundreds, 5 tens, and 8 ones and add 1 more hundred, 6 more tens, and 7 more ones, you get 3 hundreds, 11 tens, and 15 ones. Three hundred and eleven-ty fifteen is absolutely correct.

But the problem is that society doesn't let us speak this way. How could we fix up this answer so that others can understand? We can do some explosions!

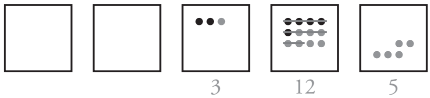
For example, 10 dots can explode from the group of 15, leaving 5 dots behind and adding a dot 1 place to the left. We now have 3|12|5.

6.16

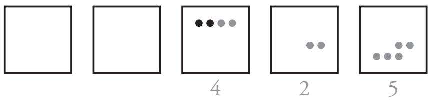


We can explode again, from the pile of 12.

6.17



$$\begin{array}{r} 258 \\ +167 \\ \hline 3\,12\,5 \end{array}$$



$$\begin{array}{r} 258 \\ +167 \\ \hline 425 \end{array}$$

We see the answer, 425. The world understands what we are saying now!

So, why are we taught to conduct long additions from right to left in school when working the other way is fine? Let's look at the right-to-left school approach.

6.18

$$\begin{array}{r} {}^{1}258 \\ +167 \\ \hline 5 \end{array} \qquad \begin{array}{r} {}^{1\,1}258 \\ +167 \\ \hline 25 \end{array} \qquad \begin{array}{r} {}^{1\,1}258 \\ +167 \\ \hline 425 \end{array}$$

Starting at the right, we have: 8 + 7 = 15, but we don't write 15. Instead, we write 5 and add 1 to the tens column. That's an explosion.

Then, 1 + 5 + 6 = 12, but we only write 2 and explode another 10, adding an extra 1 to the hundreds column. And then, 1 + 2 + 1 makes 4 hundreds. We again see the answer, 425.

It's all just a style thing. Some people find it easy to do all of their work from left to right and then do all of their explosions at the end. The traditional algorithm goes from right to left and has you attend to all the explosions as you goes along. And explosions are usually called carries in this method. Either way is correct.

1 ← 10 MULTIPLICATION

What is 243,192 × 4?

6.19

2	4	3	1	9	2

What is the answer, using a 1 ← 10 machine?

We're being asked to quadruple this number, so let's quadruple it!

We have 2 one hundred thousands. Quadrupling must give us 8 of them.

We have 4 ten thousands. Quadrupling must give us 16 of them.

3 thousands quadrupled makes 12 thousands.

1 one hundred quadrupled is 4 hundreds.

9 tens quadrupled is 36 tens.

2 ones quadrupled makes 8 ones.

6.20

8	16	12	4	36	8

The answer to the problem is 8|16|12|4|36|8, eight hundred and sixteen-ty twelve thousand, four hundred and thirty-six-ty, eight.

6.21

We can do the explosions to fix this answer for the rest of the world if we want to: 972,768.

8	16	12	4	~~3~~6	8
8	16	12	7	6	8
8	16	~~1~~2	7	6	8
8	17	2	7	6	8
8	~~1~~7	2	7	6	8
9	7	2	7	6	8

1 ← 10 SUBTRACTION

Subtraction is just the addition of the opposite. This means that we need to work with opposites in a 1 ← 10 machine—that is, we need to work with the opposite of dots. Previously, we called the opposite of a dot an anti-dot and drew anti-dots as hollow circles.

6.22

1 + –1 = 0

Recall that a dot and anti-dot together annihilate one another to disappear, just like matter and antimatter (**figure 6.22**).

6.23

$$\begin{array}{r} 423 \\ \underline{-254} \end{array}$$

At left is a traditional subtraction problem, 423 – 254, lined up by columns.

Let's go from left to right: 4 – 2 = 2, 2 – 5 = –3 and 3 – 4 = –1.

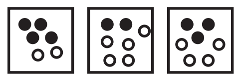
6.24

$$\begin{array}{rrrr} & 4 & 2 & 3 \\ - & 2 & 5 & 4 \\ \hline & 2 & -3 & -1 \end{array}$$

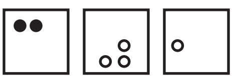
The answer is two hundred, negative three-ty, negative one.

6.25

Here's the dots-and-boxes picture: 4 hundreds and 2 anti-hundreds, 2 tens and 5 anti-tens, and 3 ones and 4 anti-ones.



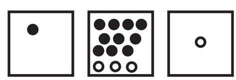
6.26



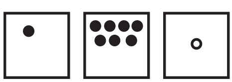
After annihilations, we are left with 2 hundreds, 3 anti-tens, and 1 anti-one—or two hundred, negative three-ty, negative one.

How can we fix this up for the rest of the world to understand? We could unexplode 1 of the dots in the hundreds place. After all, it came from 10 solid dots in the box before it. (See **figure 6.27**.)

6.27



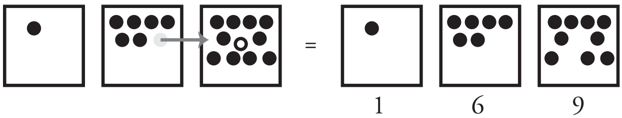
6.28

 = 1 7 –1

After some annihilations, we have 7 dots in the tens place. We have the answer, one hundred and seventy negative one.

This is still a bit strange. Next, we can unexplode a tens dot to produce 10 dots in the ones place.

6.29



We see the answer, 169, appears. The world understands that answer!

The traditional subtraction algorithm we were taught in school uses unexplosions, too. Unexploding is called borrowing there.

6.30

$$\begin{array}{r} 4\overset{1}{\cancel{2}}\,{}^{1}3 \\ -254 \\ \hline 9 \end{array} \qquad \begin{array}{r} \overset{3}{\cancel{4}}\,{}^{1}\overset{1}{\cancel{2}}\,{}^{1}3 \\ -254 \\ \hline 69 \end{array} \qquad \begin{array}{r} \overset{3}{\cancel{4}}\,{}^{1}\overset{1}{\cancel{2}}\,{}^{1}3 \\ -254 \\ \hline 169 \end{array}$$

Start from the right. What is 3 – 4? This can't be done. Unexplode 1 of the dots from the tens place and make the 3 a 13. We now have 13 – 4 = 9.

In the tens place, we now have 1 – 5, which can't be done. So, we unexplode from the hundreds place and make it 11 – 5 = 6.

Finally, in the hundreds place, we have 3 – 2, which we can do. It is 1.

The answer, 169, appears.

So, again, this is a style thing. You can be quirky and go from left to right and do the unexplosions at the end, or you can go from right to left and unexplode as you go along. Either way is correct.

FURTHER EXPLORATION

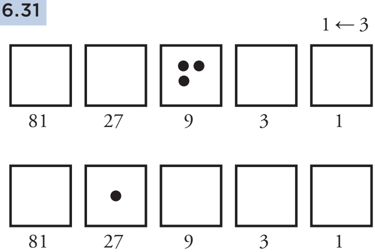
WEB

Tanton, "Exploding Dots."
http://gdaymath.com/courses/exploding-dots/

PROBLEMS

1. In a 1 ← 3 machine, 3 dots in any 1 box explode away and are replaced by a single dot in the box 1 place to their left.

6.31



a What is the code for 21 in a 1 ← 3 machine?

b What is the code for 100?

c Which number has code 21021 in the 1 ← 3 machine?

2. To multiply a number by 10 in our system of base-10 arithmetic, we might be told to just add a 0 to the number. For example, 123 × 10 = 1230.

Explain why, despite the dubious language, adjoining a 0 to a number written in base 10 does indeed correspond to multiplication by 10.

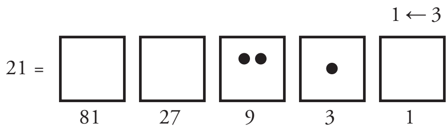
3. Many words in the English language have prefixes that refer to counts. For example, a monologue ("mono-") is a speech by 1 person, as opposed to a dialogue ("dia-") between 2 people. A tricycle ("tri-") has 3 wheels, and a student might ride that tricycle around a campus quadrangle ("quad-"). Give some examples of words with prefixes linked to the counts of 5 through 10.

4. Why was the 50th Super Bowl of 2016 not designated in Roman numerals?

SOLUTIONS

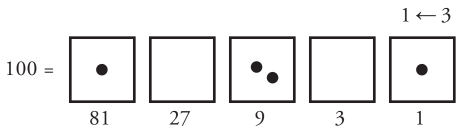
1. **a** 21 has code 210.

6.32



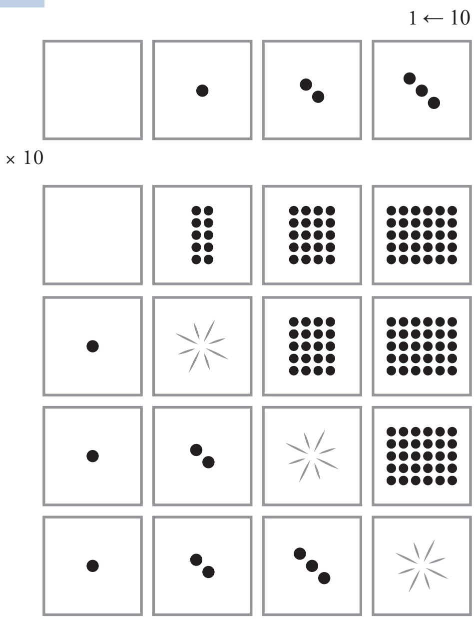
b 100 has code 10201.

6.33



c The number with code 21021 is (2 × 81) + (1 × 27) + (0 × 9) + (2 × 3) + (1 × 1) = 196.

6.34



2. In a 1 ← 10 machine, multiplying the count of dots in a box by 10 causes explosions, each placing a dot 1 place to the left. In net effect, it appears that each digit shifts 1 place to the left (which is the same effect as seeing a 0 tagged onto the end of the number).

3. For example, the Pentagon is the name of the 5-sided building in Washington DC; a hexagon is a 6-sided polygon; September is the name of the month that was, at one point, the seventh month of the year; an octopus is a creature with 8 legs; a nanometer is one billionth (10^{-9}) of a meter; and a decade is a set of 10 years.

4. The general populous is no longer familiar with Roman numerals. The number 50 is represented by the symbol L, and writing "Super Bowl L" was considered strange.

   Roman numerals do not follow a system of place value and are awkward for doing arithmetic. (Care to compute XVI + XLXI + XXC + XXVII without translating to our Hindu-Arabic place-value system?) The Roman numeral system was abandoned in Europe in the early 1200s.

PUSHING LONG DIVISION TO NEW HEIGHTS

LECTURE 7

Previously, you learned about basic arithmetic in a $1 \leftarrow 10$ machine. You examined addition, subtraction, and some multiplication, and you made sense of the standard school algorithms. This lecture will delve into the fourth piece of arithmetic: division. You might remember learning the standard long-division algorithm in school and being confused by it. This lecture will finally make sense of it. The $1 \leftarrow 10$ machine will explain everything, and the dots-and-boxes method will help.

STANDARD LONG DIVISION

7.1

$12\overline{)276}$

The standard long-division algorithm that you were probably taught as a child is confusing. For example, let's divide 276 by 12 using long division. We start by writing the problem as shown in **figure 7.1**.

Then, the algorithm goes as follows: Look at the number 276 and read it left to right. (Notice that we're using left to right all of a sudden. All the other algorithms we were taught—long addition, subtraction, long multiplication—had us start right to left.)

How many times does 12 go into the first number 2? It doesn't, so move on to the second digit and now think 27. (The number is 276, not 27. Why look at just 27?)

How many times does 12 go into 27? It goes twice, so you write 2 at the top. You might have also been taught to write 2 × 12 = 24 and perform a subtraction: 27 – 24 gives 3.

You might have then been required to draw an arrow to bring down the 6 and magically change 3 into 36.

7.2

$$\begin{array}{r} 23 \\ 12\,\overline{)\,276} \\ -24 \\ \hline 36 \\ -36 \\ \hline 0 \end{array}$$

How many times does 12 go into 36? The answer is 3 times. We write 3 at the top and also write 3 × 12 = 36. Do a subtraction and get 0, which is apparently good, and then we see the final answer, all at the top: 23.

If you look at this algorithm at face value, it comes across as truly bizarre. But the 1 ← 10 machine saves the day! By doing division in dots and boxes, we can perform this work of division with natural ease and see clearly why this algorithm works.

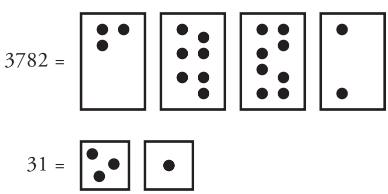
And we can just do the long division without having to memorize the algorithm. We can follow our common wits in mathematics, rather than memorizing formulas and procedures and trying to keep them all straight.

DOTS-AND-BOXES DIVISION

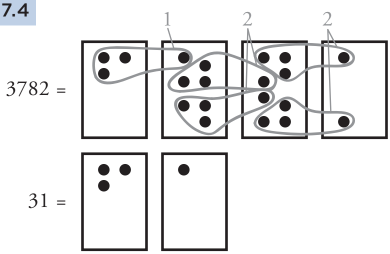
What is 3782 divided by 31?

Figure 7.3 shows what 3782 and 31 look like in dots and boxes.

7.3



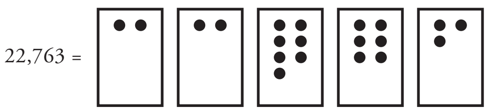
7.4



Do we see any 31s in the picture of 3782? There are several: 1 at the hundreds level, 2 at the tens level, and 2 at the ones level. Let's circle them. (Remember to take note of exactly where all 31 dots lie if you were to do the unexplosions in each group.)

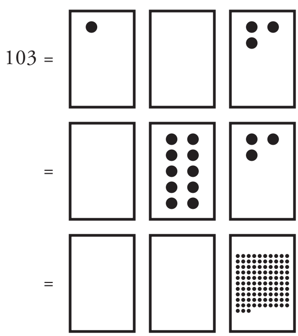
We see that the answer is 122: 3782 divided by 31 is 122.

7.5



What is 22,763 divided by 103?

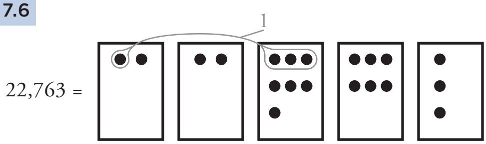
Start by drawing 22,763 and 103.



Notice that 103 is tricky. It is 1 dot, blank space, 3 dots. So, in which box do all 103 dots actually lie? If we unexplode the 1 dot in the hundreds place, we get 10 dots in the tens place, which unexplode again to make 103 dots in the ones place. So, in this picture of 1 dot, no dot, 3 dots, all 103 dots are actually sitting at the rightmost level.

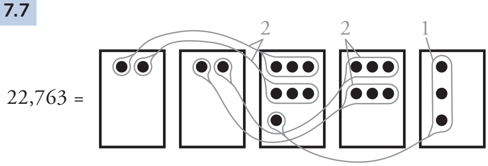
Keeping that in mind, do we see any 103s in our picture of 22,763?

7.6



There is at least 1. Remember that we're looking for 1 dot, blank, 3 dots. And keep in mind where all 103 actually lie.

7.7



Can you see more? Let's circle them. In total, there are 2 at the hundreds level, 2 at the tens level, and 1 at the ones level. The answer is 221. 22,763 divided by 103 is 221.

What is 4696 divided by 21?

7.8

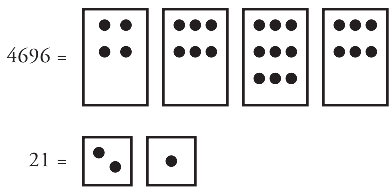
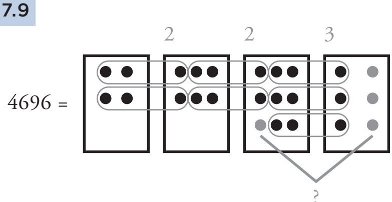


Figure 7.8 shows 4696 and 21 in dots and boxes:

Do we see any 21s in the picture for 4696? There are lots of them! Let's circle them.

There are 2 at the hundreds level, 2 at the tens level, and 3 at the ones level. But we have a problem: There are still dots left over—1 ten and 3 ones.

7.9



This problem is incomplete because there are dots left over that we are failing to divide.

Actually, this is not uncommon in division. These are called remainders.

We actually have 13 dots left over—1 ten and 3 ones—so we have just shown that 4696 divided by 21 is 223 with a remainder of 13. This is written as follows.

$$4696 \div 21 = 223 \text{ R } 13$$

So, even if there is a remainder in a division problem, the $1 \leftarrow 10$ machine shows you what it is.

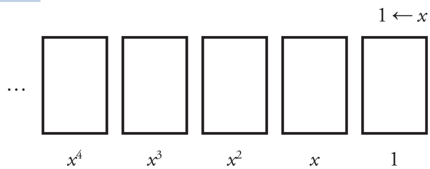
1 ← X MACHINE DIVISION

All the division we have done so far has been based on a $1 \leftarrow 10$ machine. But it can all be repeated, without a hitch, in any machine we like: a $1 \leftarrow 2$ machine, a $1 \leftarrow 7$ machine, or a $1 \leftarrow 92$ machine.

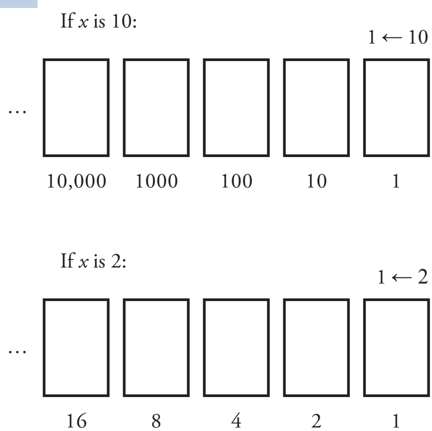
So, let's do division in all machines at once. In other words, let's perform division in some machine, without knowing which machine we are in.

In math class, it is standard to label a quantity whose value you don't know as x. So, let's follow that tradition and do division in an $1 \leftarrow x$ machine, without knowing the value for x.

7.10



7.11



But we do know that in an $1 \leftarrow x$ machine, dots in the rightmost box, as always, are worth 1.

And x of those make 1 dot 1 box over. Dots in the next box over are thus worth x 1s—that is, x. And x x's make the next box over x^2. And x x^2s make the next box over x^3—and so on.

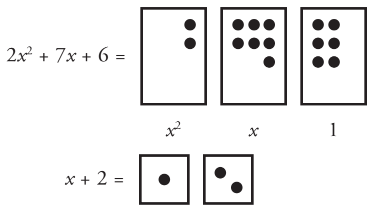
If you discover that x was 10 all along, then we have the values $1, 10, 10^2 = 100, 10^3 = 1000, \dots$ on the bottom, just as we would expect for a $1 \leftarrow 10$ machine.

If you discover instead that x is 2, then we have a $1 \leftarrow 2$ machine with the values 1, 2, 4, 8, 16, … on the bottom.

But the problem is that you don't know what x is. Nonetheless, try to solve the following visually scary division problem from advanced algebra, with no help or guidance:

$$(2x^2 + 7x + 6) \div (x + 2)$$

7.12

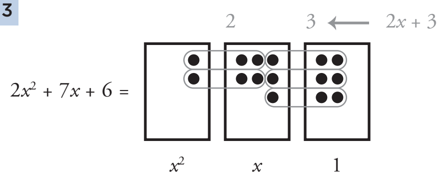


We're in an $1 \leftarrow x$ machine. $2x^2 + 7x + 6$ reads as 2 x^2s, 7 x's, and 6. It looks like **figure 7.12**.

So, $x + 2$ looks like 1 dot next to 2 dots.

This is division. So, we're looking for patterns of 1 dot next to 2 dots in a picture of 2-7-6. Do we see any? Yes, there are lots of them! Let's circle them.

7.13



We see 2 at the x level and 3 at the ones level. The answer is $2x + 3$.

$$(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$$

Suppose that x really was 10 all along—that this was a $1 \leftarrow 10$ machine. What have we just computed?

$2x^2 + 7x + 6$ is 2 hundreds, 7 tens, and 6 ones, which is 276.

$x + 2$ is $10 + 2$, which is 12.

And we got the answer, $2x + 3$, which is 2 tens and 3, or 23.

For $x = 10$:
$2x^2 + 7x + 6 = 2 \times 100 + 7 \times 10 + 6 = 276$
$x + 2 = 10 + 2 = 12$
$2x + 3 = 2 \times 10 + 3 = 23$

We've just computed that 276 divided by 12 is 23.

Suppose instead that x was actually 2 all the time—that this is a $1 \leftarrow 2$ machine.

For $x = 2$:
$2x^2 + 7x + 6 = 2 \times 4 + 7 \times 2 + 6 = 28$
$x + 2 = 2 + 2 = 4$
$2x + 3 = 2 \times 2 + 3 = 7$

$2x^2 + 7x + 6$ is 2 fours, 7 twos, and 6, which is $8 + 14 + 6 = 28$.

$x + 2$ is $2 + 2 = 4$.

And we got the answer: $2x + 3$ is $(2 \times 2) + 3 = 7$.

We've just computed that 28 divided by 4 is 7—which is correct!

In fact, no matter which number we choose for x, we've just computed a valid division problem. We've just done an infinite number of arithmetic problems all at once—one for each value of x you can imagine!

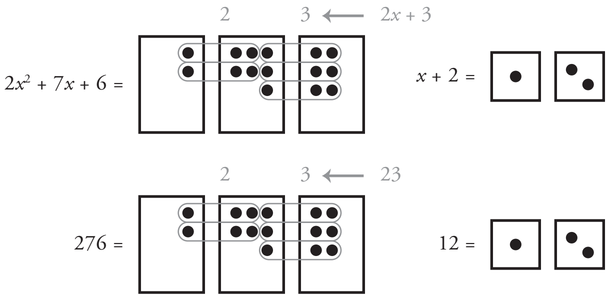
Expressions like this involving powers of an unknown are called polynomials, which are just numbers written in base x rather than base 10.

Mathematicians like to work with the most general base to see how properties of arithmetic apply to all systems simultaneously. It would be inefficient to study $1 \leftarrow 2$ machines separately from $1 \leftarrow 10$ machines when they are coming from the same general $1 \leftarrow x$ system.

Unfortunately, it is not always pointed out to high school students that all the work they do on polynomials in algebra class is mostly a repeat of all their grade school arithmetic, just done in base x rather than base 10.

In fact, look at the dots-and-boxes pictures for $(2x^2 + 7x + 6) \div (x + 2)$ in base x and $276 \div 12$ in base 10. They are the same!

7.14



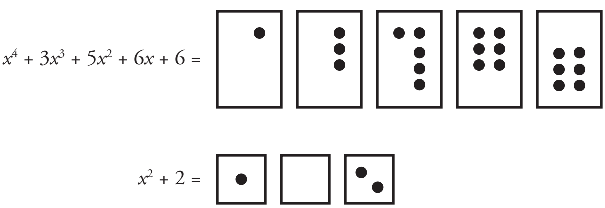
Polynomial work in upper algebra is grade school arithmetic in disguise.

Let's see how natural polynomial division really is with another, even scarier-looking example.

Here's the setup: $x^4 + 3x^3 + 5x^2 + 6x + 6$ and $x^2 + 2$.

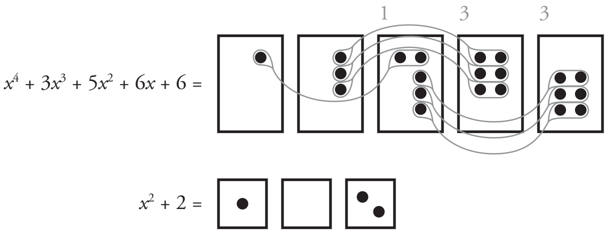
$$\frac{x^4 + 3x^3 + 5x^2 + 6x + 6}{x^2 + 2}$$

7.15



We're looking for patterns of 1 dot, space, 2 dots. Let's circle them.

7.16



We have 1 at the x^2 level, 3 at the x level, and 3 at the ones level. The answer is $x^2 + 3x + 3$.

$$\frac{x^4 + 3x^3 + 5x^2 + 6x + 6}{x^2 + 2} = x^2 + 3x + 3$$

And if x is really 10 throughout all of this, what division problem in arithmetic have we just conducted?

$13{,}566 \div 102 = 133$.

FURTHER EXPLORATION

WEB

Tanton, "Exploding Dots."
http://gdaymath.com/courses/exploding-dots/

PROBLEMS

1. **a** Compute $2765 \div 11$ using a $1 \leftarrow 10$ machine.

b Compute $3900 \div 12$ using a $1 \leftarrow 10$ machine.

2. Compute $(2x^4 + 3x^3 + 8x^2 + 5x + 6)/(x^2 + x + 2)$ in a $1 \leftarrow x$ machine.

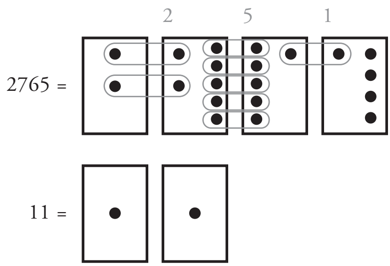
3. Using a $1 \leftarrow 5$ machine, show that, in base 5, $2003 \div 11 = 132$ with a remainder of 1.

4. A 3-digit number has the property that its middle digit equals the sum of the other 2. Why is this number sure to be divisible by 11?

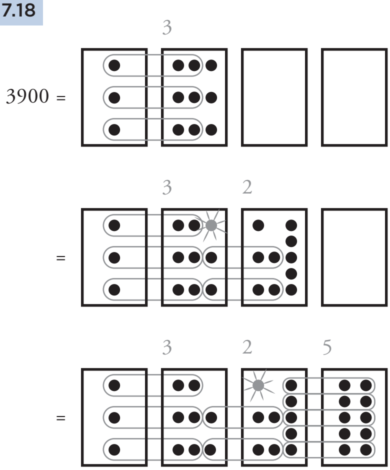
SOLUTIONS

1. **a** $2765 \div 11 = 251 \text{ R } 4$

7.17



7.18



b $3900 \div 12 = 325.$

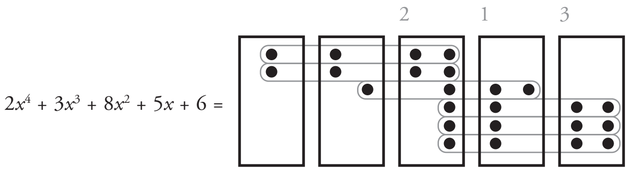
Start by identifying groups of 12.

Unexplode to identify more groups of 12.

And unexplode 1 more time.

2. $(2x^4 + 3x^3 + 8x^2 + 5x + 6)/(x^2 + x + 2) = 2x^2 + x + 3.$

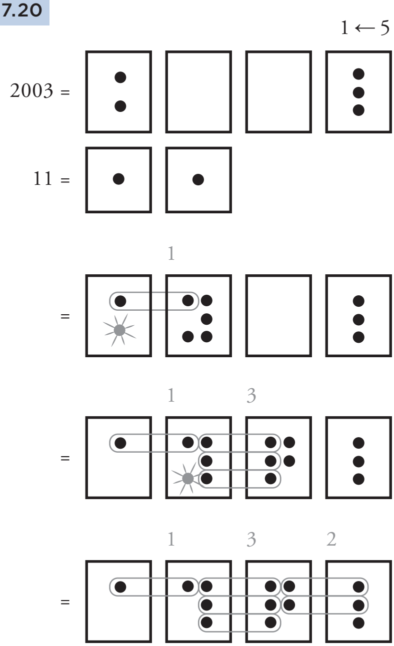
7.19



3. This time, 1 dot unexplodes to become 5 dots.

We do indeed see that $2003 \div 11 = 132$ R 1.

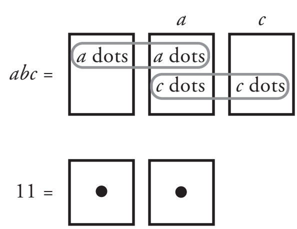
7.20



Note: In base 5, 2003 is the number $(2 \times 5^3) + (0 \times 5^2) + (0 \times 5) + (3 \times 1) = 253$; 11 is $5 + 1 = 6$; and 132 is $(1 \times 5^2) + (3 \times 5) + (2 \times 1) = 42$, and we have just computed $253 \div 6 = 42$ R 1 as a base-10 statement.

4. Write the 3-digit number as abc. We are told that $b = a + c$. We then see that $abc \div 11 = ac$.

7.21



PUSHING LONG DIVISION TO INFINITY

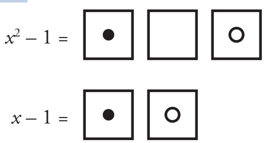
LECTURE 8

You have learned about long division in unknown base x, which turned out to be long division in every possible base, all done simultaneously. This perspective shows that the polynomial algebra of advanced school mathematics is a natural extension of familiar grade school arithmetic. Dots and boxes makes this apparent. But does this dots-and-boxes division method work for all polynomials? In this lecture, once you learn how to do polynomial division for negative numbers, you will go beyond high school algebra to new mathematical wonders—and to the infinite.

NEGATIVE NUMBERS

What is $(x^2 - 1) \div (x - 1)$?

8.1



What does $x^2 - 1$ look like? It is 1 dot in the x^2 box and 1 anti-dot in the units box.

And we're dividing by $x - 1$, which is 1 dot right next to 1 anti-dot.

Do we see 1 dot next to an anti-dot in this picture? No. It seems we're stuck.

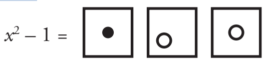
So, maybe this dots-and-boxes method doesn't always work after all!

Is there anything we can do? We do see that single dot to the left. We could try unexploding it to get some more dots in the picture.

That's a great idea, except in this $1 \leftarrow x$ machine, we don't know what x is. If it's a $1 \leftarrow 10$ machine, then we know that we would need to draw 10 dots when we unexplode. But if it is a $1 \leftarrow 2$ machine, then we would need to draw 2 dots when we unexplode. But because we don't know what x is, we don't know how many dots to draw when we unexplode. This idea is of no help.

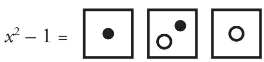
So, it looks like we are truly stymied. The problem is that we want to see 1 dot next to an anti-dot, and we don't see any. If there is something in life you want, then just make it happen (and deal with the consequences!). We would love to see an anti-dot right next to the single black dot to the left. So, let's just make it happen! Let's put an anti-dot in the picture where we want one.

8.2



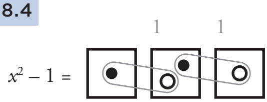
But we can't just change the question—that second box is meant to be empty. To counter our action, we need to put in an actual dot, as well. That keeps the box empty, because a dot and an anti-dot annihilate.

8.3



Now we see 2 groups of what we are looking for.

8.4



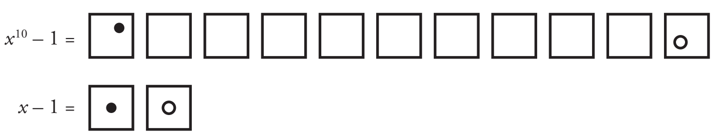
We see that $(x^2 - 1) \div (x - 1)$ is 1 x and 1 one: $x + 1$.

$$\frac{x^2 - 1}{x - 1} = x + 1$$

DOT/ANTI-DOT PAIRS

Let's find $(x^{10} - 1) \div (x - 1)$. Here's a picture of the setup:

8.5



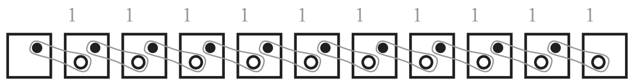
Let's add a whole slew of dot/anti-dot pairs, keeping all of the formerly empty boxes empty.

8.6

$x^{10} - 1 =$ 

Now we can see lots of dot/anti-dot pairs.

8.7

$x^{10} - 1 =$ 

$$\frac{x^{10} - 1}{x - 1} = x^9 + x^8 + x^7 + \ldots + x^2 + x + 1.$$

In fact, we can see that x raised to any power minus 1 can always be divided by $x - 1$.

$$\frac{x^{\text{any number}} - 1}{x - 1} = \text{something}$$

In other words, $x^n - 1$, for any number n, is a multiple of $x - 1$.

$$x^n - 1 = (x - 1) \times \text{something}$$

People seem to forget that, in algebra, x can actually be a number!

If we put $x = 44$ and raise it to the fifth power, then our formula says that $44^5 - 1$ is a multiple of $x - 1$, which in this case is $44 - 1$, or 43.

$$44^5\ (x^2 - 1) = 43 \times \text{something}$$

According to a calculator, $44^5 - 1 = 164{,}916{,}223$.

It is not obvious that this large 9-digit number is a multiple of 43—yet we just proved that it is.

PRIME NUMBERS

Is $32^{100} - 1$ prime?

Recall that a number bigger than 1 is said to be prime if its only factors are 1 and itself. For example, 10 is not prime because it has the factors 1 and 10, or 1 and itself, but also has more factors, 2 and 5. On the other hand, 11 is prime. It only has the factors 1 and 11.

So, is $32^{100} - 1$ prime? Does it have any factors other than 1 and itself?

$$32^{100} - 1 = 3273390607896\ldots$$

The number 32 multiplied by itself 100 times, or 32 raised to the 100th power, gives a huge value. It is a value 151 digits long. Subtract 1 and it is still 151 digits long. How could we possibly tell if it is prime?

We can show not only that it fails to be prime, but we can also give one of its factors.

We know that $x^n - 1$ is always a multiple of $x - 1$. This means that $32^{100} - 1$ is a multiple of $32 - 1$, or 31.

$$32^{100} - 1 = 31 \times \text{something}$$

$32^{100} - 1$ is not prime. It has a factor of 31.

In the same way, $173^{500} - 1$ cannot be prime (it has a factor of 172), $1003^{1003} - 1$ cannot be prime (it has a factor of 1002), and 71 to any power minus 1 is always a multiple of 70.

A problem comes, however, with the powers of 2: $2^n - 1$ is always a multiple of $2 - 1$, which is 1.

$$2^n - 1 = 1 \times \text{something}$$

So, are the powers of 2, minus 1, ever prime?

This is actually a very old question in mathematics. A 17th-century French monk named Marin Mersenne wondered about the numbers 1 less than the powers of 2. They were key to understanding some deep results in number theory. He wanted to know which ones were prime.

Here are the powers of 2, the doubling numbers:

$$2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \quad 128 \quad 256 \quad 512 \quad 1024 \quad \ldots$$

Here are the powers of 2 minus 1:

$$\begin{array}{ccccccccccc}
2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 & 1024 & \ldots \\
\underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \\
1 & 3 & 7 & 15 & 31 & 63 & 127 & 255 & 511 & 1023 & \ldots
\end{array}$$

Some of these numbers are indeed prime.

$$\begin{array}{ccccccccccc}
2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 & 1024 & \ldots \\
\underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \underline{-1} & \\
1 & 3 & 7 & 15 & 31 & 63 & 127 & 255 & 511 & 1023 & \ldots \\
 & \uparrow & \uparrow & & \uparrow & & \uparrow & & & & \\
 & 2^{\text{nd}} & 3^{\text{rd}} & & 5^{\text{th}} & & 7^{\text{th}} & & & &
\end{array}$$

The second one, the third one, the fifth one, the seventh one, and so on.

In fact, the second, third, fifth, and seventh positions are prime-number positions.

Mersenne noticed that prime numbers seem to occur in prime-number positions and that in each non-prime position, you get a non-prime number.

We can prove that he is right about this: 2 to a non-prime power minus 1 will never be prime.

For example, let's look at $2^9 - 1$. 9 is a composite number; 9 is 3 times 3.

$$\begin{aligned} 2^9 - 1 &= (2 \times 2 \times 2)(2 \times 2 \times 2)(2 \times 2 \times 2) - 1 \\ &= 8^3 - 1 = 7 \times \text{something} \end{aligned}$$

2^9 is the product of 9 2s, which can be grouped as 3 groups of 3 2s—$2 \times 2 \times 2$ times $2 \times 2 \times 2$ times $2 \times 2 \times 2$—and therefore be rewritten as $8 \times 8 \times 8$, which is 8^3.

Thus, $2^9 - 1$ is the same as $8^3 - 1$, which we know is 7 times something. $2^9 - 1$ is a multiple of 7, and therefore not prime.

In fact, this same idea works to show that 2 to any composite power minus 1 will always factor, and therefore not be prime.

So, is 2 raised to a prime power minus 1 sure to be prime?

$2^2 - 1$ is prime. $2^3 - 1$ is prime. $2^5 - 1$ is prime. $2^7 - 1$ is prime.

The answer is no. We happened to stop short of $2^{11} - 1$, which turns out not to be prime.

$$2^{11} - 1 = 2047 = 23 \times 89$$

But we have proved, like Mersenne said, that primes, when they occur, can only occur in prime-number positions.

Primes of this form, 1 less than a power of 2, are today called Mersenne primes. Surprisingly, as of March of 2016, only 49 examples of Mersenne primes are known. No one knows yet if there are infinitely many Mersenne primes to be found or whether there are just a finite number more to be discovered.

THE INFINITE

Let's use the $1 \leftarrow x$ machine to divide 1 by $1 - x$.

8.8

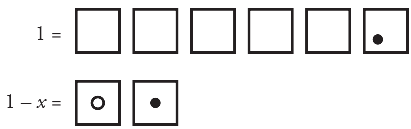
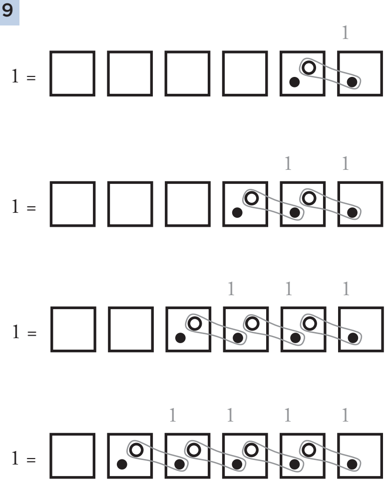


Figure 8.8 shows a picture of 1 and a picture of $1 - x$.

So, we're looking for an anti-dot next to a dot in a picture of a single dot. We don't have much to go on, so let's make it happen! Let's draw a dot/anti-dot pair.

8.9



Now we have one group of what we are looking for. But we also have an extra dot.

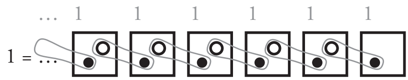
We would like to see an anti-dot next to that dot, so let's make that happen.

This gives another group and yet again an extra dot. Let's do this trick again.

Let's do it again.

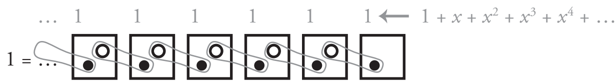
We see that we are left doing this forever.

8.10



How do we read this answer? 1 divided by $1 - x$ must equal $1 + x + x^2 + x^3 + x^4 + \ldots$, and so on forever—an infinitely long answer.

8.11



Writing it the other way around, we can say that the infinite sum $1 + x + x^2 + x^3 + x^4 + \ldots$ equals 1 over $1 - x$.

$$1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1 - x}$$

This is a very famous formula in mathematics. It is called the geometric series formula.

We have an infinite sum. We're beyond human arithmetic now. There is no way that a human being can compute the left side of this formula for any particular value of x. Humans can't sum an infinite number of numbers. Yet this formula is telling us that, if we could, we should get a concrete answer.

But can the formula actually be true?

For example, put in $x = 10$ into the geometric series formula. It seems to give something absurd. It then says that $1 + 10 + 10^2$ (100) + 10^3 (1000) + …, and so on forever, adds to 1 over (1 – 10), which is –1⁄9.

$$\text{For } x = 10$$
$$1 + x + x^2 + x^3 + x^4 + \ldots = 1 + 10 + 100 + 1000 + 10{,}000 + \ldots$$
$$\frac{1}{1-x} = \frac{1}{1-10} = \frac{1}{-9} = -\frac{1}{9}$$

There is no way that 1 + 10 + 100 + 1000 + … can, in the end, add to –1⁄9.

However, for other values of x, the formula does seem to be plausible. Try $x = ½$, for example. The formula says that 1 + ½ + ¼ + ⅛ + 1⁄16 + … should equal 2.

$$\text{For } x = \frac{1}{2}$$
$$1 + x + x^2 + x^3 + x^4 + \ldots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$
$$\frac{1}{1-x} = \frac{1}{1-½} = \frac{1}{½} = 2$$

This actually seems reasonable.

These beyond-human infinite sums belong to the world of calculus, which is all about taking matters to the beyond-human limit. It is delicate and tricky to know when certain infinite formulas like these should or shouldn't be believed. That is the work of calculus. Calculus establishes that the geometric series formula we got from dots and boxes is valid for certain values of x—namely those numbers smaller than 1 in size, such as ½.

FURTHER EXPLORATION

WEB

Tanton, “Exploding Dots.”
http://gdaymath.com/courses/exploding-dots/

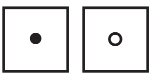
PROBLEMS

1. Compute $(x^5 - 4x^4 + 2x^3 + 7x^2 - 5x - 3) \div (x^2 - 2x - 1)$ using an $1 \leftarrow x$ machine.

2. Find 6 different factors of the number $3^{12} - 1$.

3. In a $1 \leftarrow 10$ machine, divide 21,301 by 9, but represent 9 as a dot/anti-dot pair.

8.12

9 = 

What do you notice about the answer and the following sums?

2
$2 + 1 = 3$
$2 + 1 + 3 = 6$
$2 + 1 + 3 + 0 = 6$
$2 + 1 + 3 + 0 + 1 = 7$

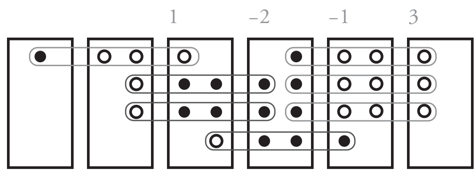
4. Compute the first few terms of $1/(1 - x - x^2)$.

SOLUTIONS

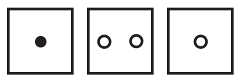
1. $(x^5 - 4x^4 + 2x^3 + 7x^2 - 5x - 3)$
$\div (x^2 - 2x - 1) = x^3 - 2x^2 - x + 3.$

8.13

$x^5 - 4x^4 + 2x^3 + 7x^2 - 5x - 3 =$



$x^2 - 2x - 1 =$



2. We have that $x^n - 1 = (x - 1) \times$ something. So:

$$3^{12} - 1 = (3 - 1) \times \text{something}$$
$$= 2 \times \text{something}$$

$$3^{12} - 1 = (3^2)^6 - 1 = 9^6 - 1$$
$$= (9 - 1) \times \text{something} = 8 \times \text{something}$$

$$3^{12} - 1 = (3^3)^4 - 1 = 27^4 - 1$$
$$= (27 - 1) \times \text{something} = 26 \times \text{something}$$

$$3^{12} - 1 = (3^4)^3 - 1 = 81^3 - 1$$
$$= (81 - 1) \times \text{something} = 80 \times \text{something}$$

$$\ldots$$

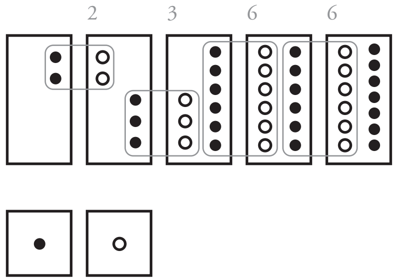
We see the factors 2, 8, 26, and 80. Because these are factors, then so are 4 (from $8 = 2 \times 4$), 13 (from $26 = 2 \times 13$), and 5 (from $80 = 5 \times 16$)—and so on.

3. In each box, we must insert a count of anti-dots that is equal to the number of dots in the previous box. This means adding the same number of dots to that box, as well. In effect, counts of dots are being added as we go along.

8.14

21,301 =

9 =



The sums—2, 2 + 1 = 3, 2 + 1 + 3 = 6, 2 + 1 + 3 + 0 = 6, and 2 + 1 + 3 + 0 + 1 = 7—match the counts of pairs we circle: 21,301 ÷ 9 = 2366 R 7.

4. $1/(1 - x - x^2)$

$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21\,x^7 + 34x^8\,x + \ldots.$

VISUALIZING DECIMALS

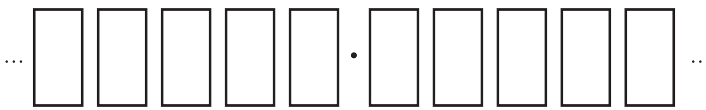
LECTURE 9

Thus far, when using the dots-and-boxes method, we have always started with a row of boxes extending as far to the left as we please. But why are there not some boxes to the right as well? Mathematicians are drawn to symmetry. They recognize it as a powerful platform for clever thinking—both forward and backward thinking—and are often perturbed by systems that don't, at first, seem symmetrical. So, in this lecture, let's be symmetrical. Let's not just have boxes to the left; let's also have boxes to the right, extending as far as we please.

BOXES TO THE RIGHT

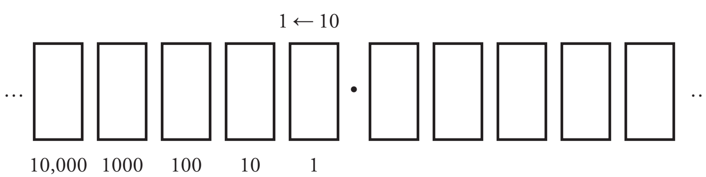
To keep track of which are the left boxes and which are the right ones, let's use a place mark—a point—to separate them.

9.1

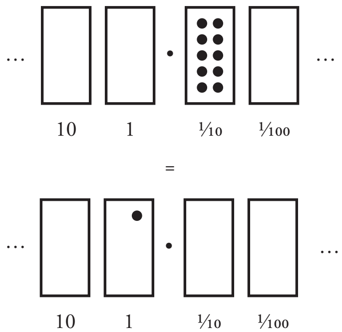


What could these rightmost boxes mean? To make sense of them, let's work with a specific dots-and-boxes machine, the $1 \leftarrow 10$ machine. In this machine, the leftmost boxes represent the powers of 10. What could the rightmost boxes represent?

9.2



9.3

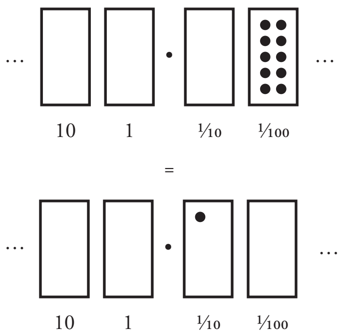


We know that 10 dots in any 1 box are equivalent to just 1 dot 1 place to the left.

So, 10 dots in the box to the right of the point is worth the same as 1 dot to the left of the point.

If 10 somethings make a dot worth 1, then it must be that each dot in the box just to the right of the point is worth one-tenth, because 10 one-tenths make 1.

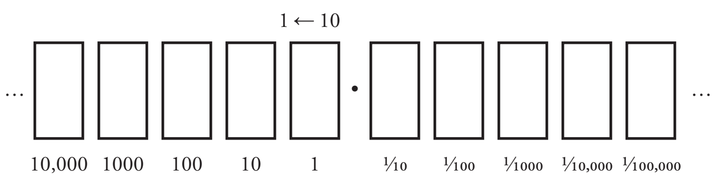
9.4



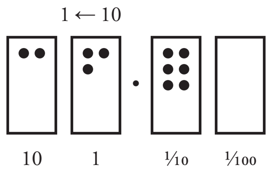
And 10 dots in the next box over is equivalent to 1 dot in the tenths box. It must be that each dot in the next box over is worth a hundredth: 10 one-hundredths make a tenth.

And dots in the next box over must be worth a thousandth: 10 one-thousandths make a hundredth—and so on.

9.5

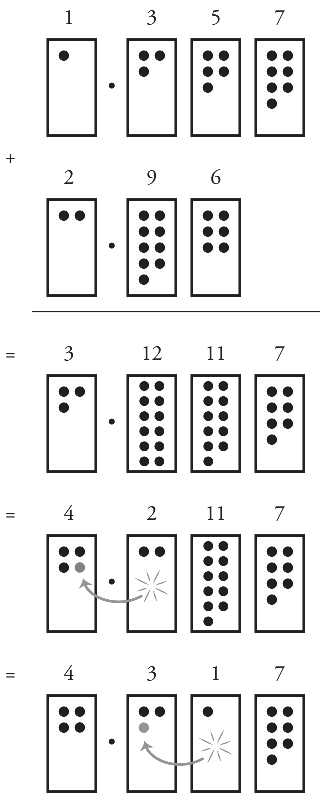


9.6



The boxes to the left of the point represent the powers of 10, and the numbers to the right represent the powers of one-tenth. In a 1 ← 10 machine, that point is called a decimal point—"deci-" means 10—and we have just discovered the means to write numbers as decimals.

9.7



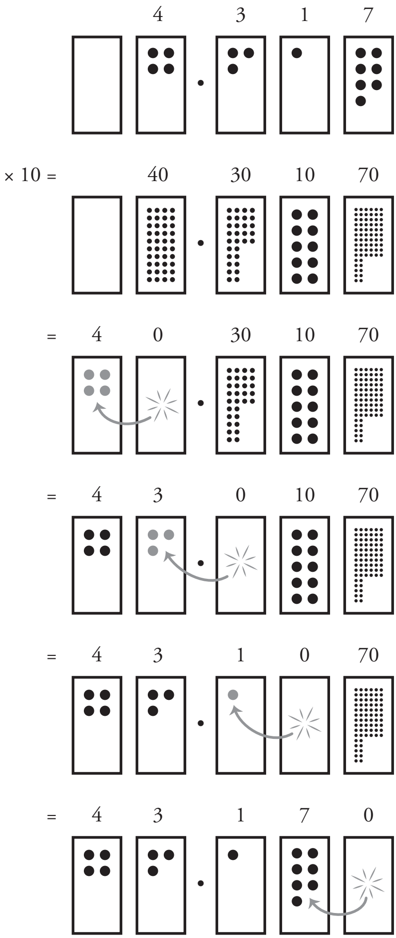
For example, **figure 9.6** is a picture of the number 23.6; it's 2 tens, 3 ones, and 6 tenths.

All of the arithmetic of decimals is just the same as the arithmetic we first learned for a 1 ← 10 machine. Nothing is new; we're just playing with the same ideas seen in a new light.

For example, **figure 9.7** is a picture of 1.357 + 2.96: 1 + 2 makes 3, 3 tenths + 9 tenths make 12 tenths, 5 hundredths + 6 hundredths make 11 hundredths, and 7 thousandths is still 7 thousandths. Just as previously, we can do this left to right and get an answer of 3 point twelve-ty eleven-ty 7, which people tend not to say. But, just as previously, we can fix this up for the rest of the world and do some explosions. We can make it 4 point 2 eleven-ty 7 and then 4.317.

DECIMALS

9.8



In school, you might've been taught a rule that if you multiply a decimal by 10 or 100, you had to move the decimal point a place or 2 over. It might have been confusing to try to memorize which way to move that decimal point—to the left or to the right? It was a rule that you just had to memorize, and memorizing correctly is always difficult!

So, let's look at what multiplying by 10 does to a decimal—for example, 4.317 times 10. Here's a picture of 4.317 again, but this time we are writing numbers rather than drawing dots: 4 dots, 3 dots, 1 dot, and 7 dots.

When we multiply all of the quantities by 10, we get 40 dots, 30 dots, 10 dots, and 70 dots.

Now matters are ripe for explosions! The 40 dots explode to make 4 dots 1 place to the left.

Then, 30 dots explode to produce 3 dots to the left.

Then, 10 dots explode, and finally, 70 dots explode.

We see the answer 43.17 appear—that is, 4.317 × 10 = 43.17.

It looks as though the decimal point moved 1 place to the right, but it is really that the digits 4, 3, 1, and 7 each moved 1 place to the left—because of those explosions. The rule you were taught to memorize in school is the opposite of what is actually going on!

Of course, multiplying a number just over 4 by 10 should give an answer of just over 40, so it is easy to keep track of which way the decimal point should allegedly move, but the pictures of dots and boxes gives the visual of what is really going on—and why. If matters ever get more complicated, the dots-and-boxes picture won't let you down.

FRACTIONS

Decimals often arise in our daily lives when we work with fractions. We often want to convert fractions to decimals and vice versa. For example, ½ is the same as 0.5. After all, 0.5 is 5⁄10, which is ½. And ¼ is 0.25.

We just converted a pair of easy fractions to decimals. But matters can be quite awkward. For example, what's 4⁄7 as a decimal?

We have the tools to convert all fractions into decimals. All we have to do is division in a 1 ← 10 machine.

Every fraction has an infinitely long decimal expansion with a repeating pattern—allowing for repeating zeros as a possibility. For example, ⅓ is 0.3333…, and ¼ is 0.2500000….

But is every infinitely long repeating decimal a fraction? For example, is 0.44444… a fraction? What number is 0.44444….?

Here's one way to answer that question. Start by giving the decimal a name. Let's call it F for Frederica.

$$F = 0.44444\ldots.$$

What can we do with Frederica? Previously, we were multiplying decimals by 10, so let's do that here. And we know what multiplication by 10 does to decimals—it moves the decimal point. (Remember that it's really a whole lot of exploding dots, but the end effect is that the decimal point seems to move.)

10 Fredericas equals 4.4444….

$$10F = 4.44444\ldots.$$

This move was inspired! Look at 4.444…. It's 4 plus 0.4444….

$$10F = 4 + 0.44444\ldots.$$

And we recognize 0.44444… in this equation as Frederica: $10F = 4 + F$.

Subtracting F from both sides gives $9F = 4$.

Dividing by 9 shows that $F = \frac{4}{9}$.

The repeating infinite decimal 0.444444… is the fraction $\frac{4}{9}$.

This technique works in general. With it, you can indeed show that any infinite decimal that repeats—either right off the bat or after an initial hiccup (such as 0.25333333…)—does equal a number that is a fraction. The converse is also true: Every fraction corresponds to a decimal with a repeating pattern, if we include repeating zeros as a permissible pattern.

IRRATIONAL NUMBERS

We have pinned down all of the numbers that are fractions by looking at decimal expansions. Those numbers with repeating decimals are precisely the fractions.

What about numbers that have infinitely long decimal expansions and do not fall into a repeating pattern?

Consider, for example, this number: 0.1011011101111011111011111110….

This is a number that is a bit bigger than 0.1, one-tenth. It is smaller than 0.2, two-tenths. So, it is some value between 0.1 and 0.2. It is somewhere on the number line.

Although the decimal expansion of this number has a pattern, it is not a repeating pattern. From this lecture's work, this means that this number cannot be a fraction!

We have just discovered a number on the number line that is not a fraction. Fractions are often called rational numbers; they are the ratio of 2 whole numbers. We now have an example of an irrational number, a number that is not rational.

There are many more examples of irrational numbers. Any number that has an infinite decimal expansion that fails to fall into a repeating pattern is an irrational number.

Proving that numbers really are irrational is usually extraordinarily difficult. For millennia, scholars wondered if the number π is a fraction. It wasn't until 1761 that Swiss mathematician Johann Lambert finally proved that π is indeed an irrational number. It was a difficult proof. And even though his proof has been revised, reworked, and simplified to some degree in the 2.5 centuries since, it is still too complicated an explanation to share in any school class or in most university classes.

FURTHER EXPLORATION

WEB

Tanton, "Exploding Dots."
http://gdaymath.com/courses/exploding-dots/

READING

Tanton, *Thinking Mathematics! Vol. 1.*

PROBLEMS

1. **a** Use a $1 \leftarrow 10$ machine to compute the decimal expansion of $4/11$.

b Which fraction has a decimal expansion of $0.2363636\ldots = 0.2\overline{36}$?

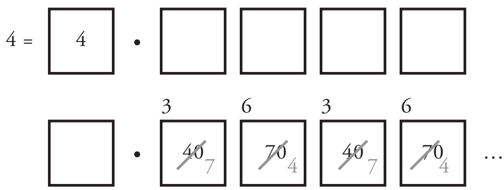
2. Is 0.45676500375281103862… rational or irrational?

3. If $x = 0.123412341234\ldots = 0.\overline{1234}$ and $y = 0.987987987\ldots = 0.\overline{987}$, must $x + y$ have a decimal expansion that repeats? Must $x \div y$?

4. For a fraction a/b, with a and b positive whole numbers, the denominator b has only 2 and 5 as prime factors. Explain why a/b therefore has a finite decimal expansion.

SOLUTIONS

1. **a** We have that $4/11 = 0.363636\ldots = 0.\overline{36}$.



$$\frac{4}{11} = 0.\overline{36}$$

Note: By inserting dots and anti-dots, we can legitimately argue that $4/11 = 0.4 \mid -4 \mid 4 \mid -4 \mid \ldots$. With unexplosions, this equals 0.363636….

b Let $F = 0.2363636\ldots$. Then, $10F = 2.363636\ldots = 2 + 4/11$. That is, $10F = 22/11 + 4/11 = 26/11$, so $F = 26/(10 \times 11) = 13/55$.

2. This is a trick question! There is not enough information to say whether or not this number is rational. Maybe the decimal expansion eventually repeats, or maybe it doesn't. We cannot tell.

3. Because each of x and y have repeating decimal expansions, they are each fractions. Thus, $x + y$ and $x \div y$ are fractions and therefore, too, have repeating decimal expansions (but we don't care to work out what they are!).

4. Consider a fraction of the form $a/(2 \times 2 \times 2 \times 2 \times 5 \times 5)$, for example. Multiplying the numerator and denominator each by another pair of 5s gives:

$$(a \times 5 \times 5)/(2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 5)$$
$$= (a \times 5 \times 5)/(10 \times 10 \times 10 \times 10)$$
$$= (a \times 5 \times 5)/10{,}000.$$

The fraction is some number of ten thousands and therefore can be expressed in just 4 decimal places.

This idea shows that any fraction with a denominator composed solely of 2s and 5s can be rewritten with a denominator that is a power of 10. It is thus represented with a finite number of decimal places.

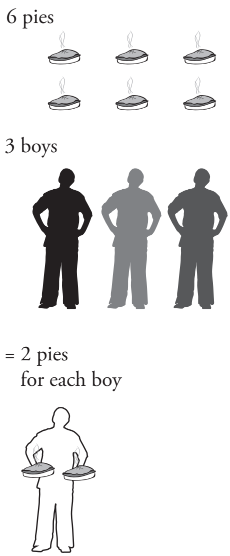
PUSHING THE PICTURE OF FRACTIONS

LECTURE 10

Through our play with infinite decimals, we have discovered that not every number is a fraction. All fractions have decimal expansions that fall into a repeating pattern. Therefore, any number that has a decimal expansion that fails to repeat cannot be a fraction. It is some other kind of number. It is called an irrational number. In this lecture, you will learn more about the mathematics of fractions and irrational numbers.

OVERVIEW OF FRACTIONS

10.1



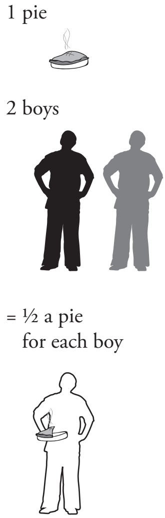
Simply put, a fraction is an answer to a division problem—that is, the answer to a sharing problem of some kind.

For example, if 6 pies are shared among 3 boys, the result is 2 pies per boy.

We write 6 over 3, and that's a fraction. It's a fraction that equals 2.

$$6/3 = 2$$

10.2



If just 1 pie is shared between 2 boys, the amount of pie that each boy receives is called a half.

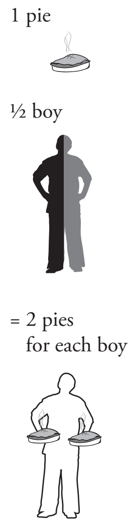
The division problem 1 over 2 is a half.

In general, the fraction a over b is the amount of pie an individual boy receives when a pies are shared equally among b boys.

What does the fraction $1/(\frac{1}{2})$ represent?

If 1 pie is distributed among half a boy, how much pie does a whole boy get?

10.3



The bottom half of the boy gets 1 whole pie, and the top half gets 1 whole pie, too. So, the whole boy gets 2 pies.

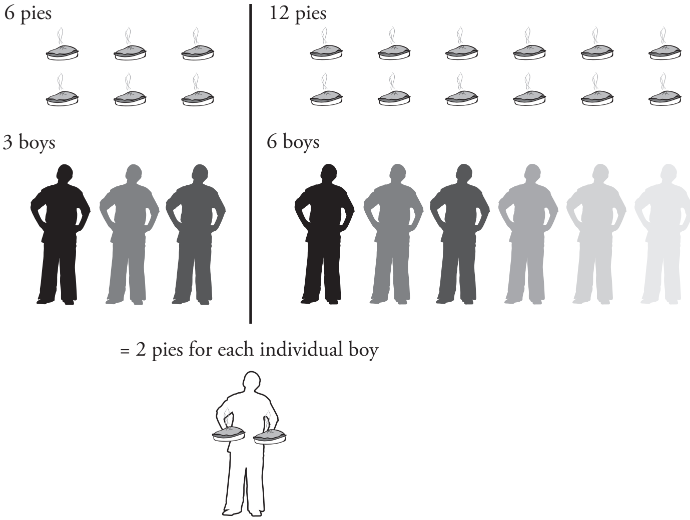
$$\frac{1}{\frac{1}{2}} = 2$$

In a sharing problem, what happens if we double the number of pies and double the number of boys? The amount of pie each boy receives remains the same. For example, in the following picture, $\frac{6}{3}$ and $\frac{12}{6}$ both give 2 pies for each boy.

Tripling the number of pies and tripling the number of boys does not change the final amount of pie per boy, nor does quadrupling the counts of each—and so on.

$$\frac{a}{b} = \frac{2a}{2b} = \frac{3a}{3b} = \dots$$

10.4



This leads to the key fraction rule, which makes the arithmetic of fractions work. Multiplying the numerator and denominator—that is, the top and bottom—of a fraction by a common value does not change the fraction.

Key Fraction Rule: $\frac{xa}{xb} = \frac{a}{b}$

This is the rule that allows us to simplify fractions. For example, for the fraction $32/20$, we can think of 32 as 4×8 and 20 as 4×5. By the key fraction rule, that common factor 4 does not matter. This fraction equals $8/5$.

$$\frac{32}{20} = \frac{4 \times 8}{4 \times 5} = \frac{8}{5}$$

10.5

$$\frac{2}{3} + \frac{1}{4}$$
$$\frac{4}{6} + \frac{2}{8}$$
$$\frac{6}{9} + \frac{3}{12}$$
$$\frac{8}{12} + \frac{4}{16}$$
$$\frac{10}{15} + \frac{5}{20}$$
$$\vdots \qquad \vdots$$

This key fraction rule also allows us to add and subtract fractions with ease. For example, to work out $2/3 + 1/4$, there are many ways that you can rewrite each of the fractions using the key fraction rule—doubling the top and the bottom, tripling the top and the bottom, quadrupling the top and the bottom, and so on.

Notice the appearance of 2 fractions with a common denominator—in this case, $8/12$ and $3/12$. Adding these 2 fractions is as easy as adding 8 apples and 3 apples. The answer is $11/12$.

10.6

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$
$$\frac{4}{6} + \frac{2}{8}$$
$$\frac{6}{9} + \frac{3}{12}$$
$$\frac{8}{12} + \frac{4}{16}$$
$$\frac{10}{15} + \frac{5}{20}$$
$$\vdots \qquad \vdots$$

In a sharing problem of *a* pies for *b* boys, how do we double the amount of pie each boy receives? We can just double the number of pies we hand out. In mathematics, this reads as follows: Doubling the amount of pie each boy receives, 2 times a/b, is achieved by sharing $2a$ pies to *b* boys.

$$2 \times \frac{a}{b} = \frac{2a}{b}$$

There is nothing special about the number 2 here. This rules shows in general how to multiply fractions by numbers.

$$\text{Fraction Rule: } x \times \frac{a}{b} = \frac{xa}{b}$$

In a problem such as $7 \times 3/7$, we have this instinctual reaction to just cross out the 7s and write 3.

$$\cancel{7} \times \frac{3}{\cancel{7}} = 3$$

This is logically correct—but the logic is long. By the multiplication rule, 7 times $3/7$ is $(7 \times 3)/7$. Now rewrite the bottom 7 as 7×1. By the key fraction rules, the 7s cancel to leave us with $3/1$, which is just 3.

$$7 \times \frac{3}{7} = \frac{7 \times 3}{7} = \frac{7 \times 3}{7 \times 1} = \frac{3}{1} = 3$$

What is 7 and $2/3$ divided by 5 and $3/4$?

Because division problems are fractions, this division problem is the following fraction:

$$\frac{7 + \frac{2}{3}}{5 + \frac{3}{4}}$$

Can we make this look more manageable? Let's use the key fraction rule and multiply the numerator and denominator each by 3. This doesn't change the fraction.

$$\frac{7 + \frac{2}{3}}{5 + \frac{3}{4}} = \frac{\left(7 + \frac{2}{3}\right) \times 3}{\left(5 + \frac{3}{4}\right) \times 3} = \frac{21 + 2}{15 + \frac{9}{4}}$$

We still have fractions within fractions. Let's now multiply the top and bottom each by 4.

$$\frac{21 + 2}{15 + \frac{9}{4}} = \frac{(21 + 2) \times 4}{\left(15 + \frac{9}{4}\right) \times 4} = \frac{84 + 8}{60 + 9} = \frac{92}{69}$$

This looks friendlier.

We've just shown that sharing 7 and ⅔ pies equally among 5 and ¾ boys gives the same result as sharing 92 pies among 69 boys.

People are usually taught a special rule for dividing fractions. Instead of memorizing a special rule, use the idea of pies and boys and the key fraction rule.

APPROXIMATING $\sqrt{2}$

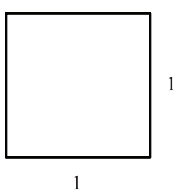
$\sqrt{2}$ is a number that, when multiplied by itself, gives the answer

$$\sqrt{2} \times \sqrt{2} = 2$$

of 2.

Actually, does such a number even exit? Is there a number that multiplies by itself to give the value 2? $1 \times 1 = 1$: too small. $2 \times 2 = 4$: too big. $1.5 \times 1.5 = 2.25$: too big. $1.4 \times 1.4 = 1.96$: too small.

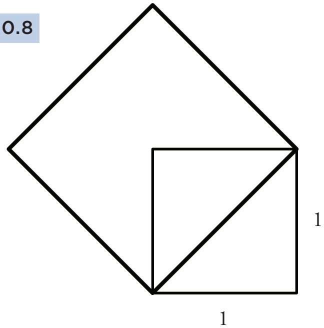
10.7



The area of a square is its side length multiplied by itself. So, to prove that $\sqrt{2}$ really exists as a number, we have to give an example of a square with an area of 2. The side length of such a square is a number that multiplies by itself to give the answer of 2.

We can certainly construct squares with an area of 1. Just use a unit length—1 inch or 1 meter, whatever your preferred unit is to get a square with an area of 1 square unit.

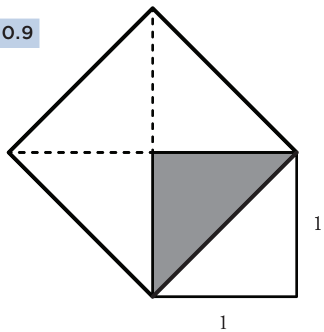
10.8



Here's how to get a square with an area of 2 from this picture: Use the diagonal as the side length of a tilted square.

Can you see that the tilted square really does have an area of 2? It helps to draw some extra lines. (See **figure 10.9**.)

10.9

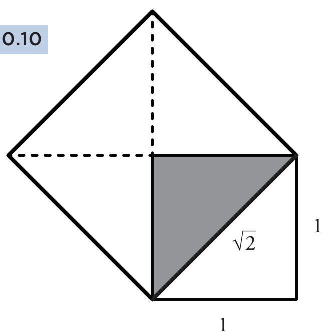


We now see that the tilted square is made of 4 triangles that each have an area of ½, and 4 halves make 2. This titled square really does have an area of 2.

This means that its side length is a number that multiplies by itself to give the answer of 2. This is $\sqrt{2}$—which does really exist. It is the diagonal of our original unit square.

On a calculator, $\sqrt{2}$ is approximately 1.4142. But this is only an approximation. For example, multiplying 1.4142 by itself gives only 1.9996164. This is really close to 2, but it is not 2.

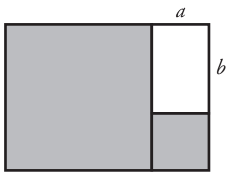
10.10



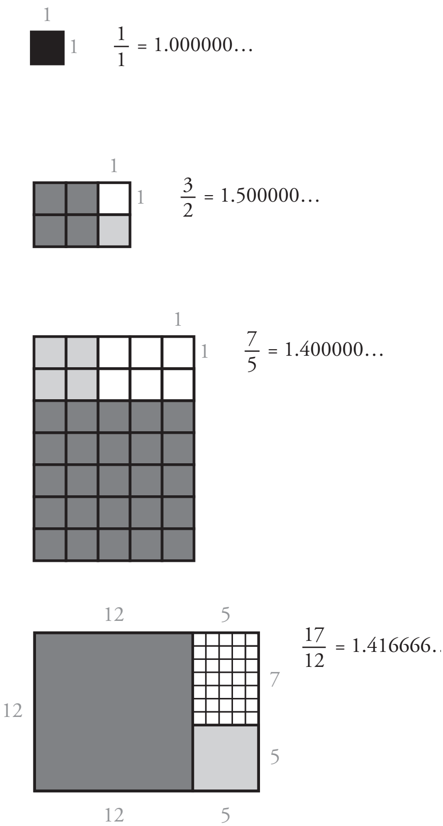
But our question is whether or not $\sqrt{2}$ is a fraction. Could it be $297/210$? Could it be $1673/1183$? Or could it be some crazy fraction with a numerator and denominator each millions of digits long?

It turns out that there is no fraction that equals $\sqrt{2}$. This number is not a fraction, but it can be approximated well by fractions.

10.11



10.12



Mathematicians have been playing with $\sqrt{2}$ for many centuries and have discovered all kinds of delightful features of it. For example, Theon of Smyrna of 1900 years ago knew that if a rectangle of sides *a* and *b* has a proportion that approximates $\sqrt{2}$, then drawing a square on the small side of that rectangle, and then a square on the long side of what results, gives a new rectangle with proportions that more closely approximate $\sqrt{2}$.

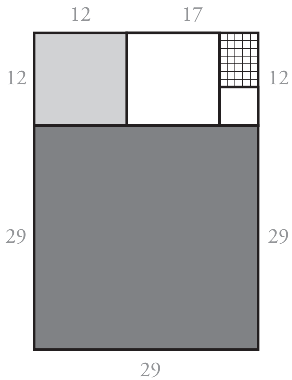
Let's say that you have a square piece of paper that is 1 unit by 1 unit. Its sides come in the ratio $1/1$, which is a lousy approximation of $\sqrt{2}$.

Now draw a square on one side, and then one on the long side of the result. This gives a 3-by-2 rectangle, and $3/2$ is 1.5, which is a better approximation of $\sqrt{2}$.

To get a better approximation still, add another square to the short side, a 2-by-2 square, and then one to the long side. This gives a 7-by-5 rectangle, and $7/5$ is 1.4. That's a better approximation of $\sqrt{2}$.

Again, add a square to the short side and then a square to the long side. This gives a 17-by-12 rectangle, and $17/12$ is about 1.41666—getting better!

10.13



$\frac{41}{29} = 1.413793\ldots$

Do it again and get a 41-by-29 rectangle, which is an even better approximation.

If you do this yet again, you get a 99-by-70 rectangle and a pretty good approximation of $\sqrt{2}$.

The sequence of fractions $\frac{1}{1}$, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, and $\frac{99}{70}$ that appeared in this process is known as Theon's ladder. These numbers arise from stacking squares and give better and better approximations of $\sqrt{2}$.

$\frac{99}{70} = 1.414285\ldots$

Stacking squares together to create larger and larger rectangles in different patterns gives approximations to different irrational numbers.

Tiling patterns with squares gives a visual way to get a handle on a whole host of irrational numbers. Although we'll never be able to view the entire infinite decimal expansion of any irrational number, we can get a handle on a square tiling: Decimal expansions have no patterns, but the tilings do. Mathematicians make great use of these tiling patterns in their work to understand properties of large classes of irrational numbers.

FURTHER EXPLORATION

WEB

Bogomolny, "The Square Root of 2 Is Irrational."
http://www.cut-the-knot.org/proofs/sq_root.shtml

Tanton, "Cool Math Essay: December 2015."
http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_December-2015_Paper-Sizes.pdf

READING

Bryant and Sangwin, *How Round Is Your Circle?*

Tanton, *Mathematics Galore!* (More on Theon's ladder.)

PROBLEMS

1. **a** Show that $3/7 \div 2/5 = 15/14$.

b Students are often told that to divide fractions, take the reciprocal of the second fraction and multiply instead.

This rule has students compute $3/7 \div 2/5$ as $3/7 \times 5/2 = 15/14$.

Explain why this rule works.

2. Compute $12/20 \div 3/4$ to show that the answer matches $(12 \div 3)/(20 \div 4)$. Is this a coincidence?

SOLUTIONS

1. **a** We have

$$\frac{\frac{3}{7}}{\frac{2}{5}} = \frac{\frac{3}{7} \times 7}{\frac{2}{5} \times 7} = \frac{3}{\frac{14}{5}} = \frac{3 \times 5}{\frac{14}{5} \times 5} = \frac{15}{14}$$

b In general:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \times b \times d}{\frac{c}{d} \times b \times d} = \frac{a \times d}{c \times b}$$

This is the same result as computing the product $a/b \times d/c$, which is the rule described.

2. We can check that $12/20 \div 3/4$ does indeed equal $4/5$.

This is not a coincidence.

The previous answer shows that $a/b \div c/d$ equals $(ad)/(bc)$. We can show that $(a \div c)/(b \div d)$ equals this as well:

$$\frac{a \div c}{b \div d} = \frac{\frac{a}{c}}{\frac{b}{d}} = \frac{\frac{a}{c} \times c \times d}{\frac{b}{d} \times c \times d} = \frac{a \times d}{c \times b}$$

So, $a/b \div c/d$ and $(a \div c)/(b \div d)$ give the same answer.

VISUALIZING MATHEMATICAL INFINITIES

LECTURE 11

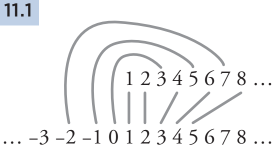
Are all sets of numbers—counting numbers, integers, rational numbers, irrational numbers—the same size? You might think that each set is infinitely big, so these sets are the same size, and that size is infinite. But in the mid-1800s, German mathematician Georg Cantor shocked the mathematical world by discovering that infinite sets need not be the same size—that there exist different sizes of infinity. For example, he discovered that the infinite size of the set of counting numbers is a different-size infinity than the size of the set of irrational numbers. In fact, he proved– that there are infinitely many different sizes of infinity. In this lecture, you will explore Cantor's logic on infinite sets.

COUNTING NUMBERS, INTEGERS, AND FRACTIONS

The set of counting numbers is a single-ended infinity, and the set of integers is a double-ended infinity.

1 2 3 4 5 6 7 8 … **smaller?**
… –3 –2 –1 0 1 2 3 4 5 6 7 8 … **larger?**

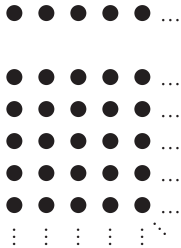
11.1



Think of the top row of numbers as dogs and the bottom row as people. Two sets are the same size if we can draw leashes so that each person is leashed to 1 dog, and each dog is leashed to 1 person. **Figure 11.1** shows that these 2 sets are the same size.

The set of integers is just as infinite as the set of counting numbers.

11.2



What about the set of fractions? What is the picture for those?

Cantor was clever and realized that the picture for the fractions is as shown in **figure 11.2**.

The single-ended infinity is the set of counting numbers, and 2-dimensional infinity is the set of all the fractions—the positive ones, at least.

11.3

1 2 3 4 5 …

1/1 1/2 1/3 1/4 1/5 …
2/1 2/2 2/3 2/4 2/5 …
3/1 3/2 3/3 3/4 3/5 …
4/1 4/2 4/3 4/4 4/5 …
5/1 5/2 5/3 5/4 5/5 …

We can put all the fractions in a 2-dimensional infinite table. The first row of the table has all the fractions with numerator 1, the second has all the fractions with numerator 2, the third has all the fractions with numerator 3, and so on. Every fraction appears in the table.

11.4

The leashing in **figure 11.4** shows that the set of all fractions is the same size as the set of counting numbers:

The first fraction to get leashed is 1/1; the second is 2/1; the third is 1/2; The fourth is 1/3; the fifth is 2/2; the sixth is 3/1; and so on.

This is one thing that people might object to: We are repeating fractions. For example, 1/1 is the number 1, as is the number 2/2, and so on. But this repetition doesn't ruin a leashing pattern. All we have to do is skip over the repeat fractions, don't bother leashing them, and just move on to the next fraction in our diagonal zigzag path and leash that. This will give a modified picture of the one we have, but it will still be a picture that leashes every fraction to a counting number.

THE SET OF ALL NUMBERS

The set of all numbers includes the fractions and the irrational numbers. This is the set of all numbers that appear on the number line.

Every fraction is a decimal that falls into a repeating pattern, and every irrational number is a decimal with no repeating pattern. The set of all the numbers on the number line is the set of all possible decimals—in fact, the set of all infinitely long decimals.

For example, the number ¼ is 0.25 as a decimal, which is the same as 0.25000000…., and the fraction ⅓ is 0.33333….
These are infinitely long decimals.

Of course, these decimal expansions are computed in base 10—that is, in a $1 \leftarrow 10$ machine. We can compute decimals in any machine we like. Here's what they look like in a $1 \leftarrow 2$ machine: ¼ is 0.01000000000…, and ⅓ is 0.0101010101….

Work in a $1 \leftarrow 2$ machine only ever uses the digits 0 and 1. If we represent 0 as a hollow dot and 1 as a black dot, then each decimal expansion corresponds to a row of black-and-white dots.

11.5

$$\frac{1}{4} = 0.010000000\ldots = \circ \bullet \circ \circ \circ \circ \circ \circ \ldots$$

$$\frac{1}{3} = 0.010101010\ldots = \circ \bullet \circ \bullet \circ \bullet \circ \bullet \ldots$$

Conversely, any row of black-and-white dots corresponds to a real number, a decimal in a $1 \leftarrow 2$ machine.

11.6

$$\circ \bullet \bullet \bullet \circ \bullet \bullet \circ \ldots = 0.01110110\ldots$$

The picture that goes with the set of all real numbers on the number line is the set of all rows of black-and-white dots. And this set is a bigger infinity than the counting numbers.

The set of all real numbers—that is, the set of all rational and irrational numbers together—is more infinite than any other set of numbers we've discussed in this lecture.

Actually, we need to more precise. We've actually only played with decimals that begin "zero point something": 0.0100000…, 0.1010101…, and so on. In other words, we have proved that the set of all real numbers between just 0 and 1 is more infinite than the set of counting numbers.

But if this tiny piece of the number line is already more infinite than the set of all the counting numbers, then it follows that the set of all the real numbers—the entire number line—is also more infinite than the set of counting numbers.

There is actually a slight problem with the argument we just presented. There is a repeat problem with the real numbers. For example, the number 1 can be represented as the infinite decimal 1.000000… or as 0.999999…. Similarly, ½ can be either 0.50000000… or 0.499999….

We have to do a bit of adjustment on our argument to handle the repeat representations. But it can be fixed.

We have 2 types of infinity: the infinity of the counting numbers, the integers, and the fractions; and the infinity of all numbers, the rational numbers and the irrational numbers together.

One question immediately comes to mind: Are there any sets that represent a third, new type of infinity? A fourth type of infinity? How many different infinities are there?

The answer is that there are infinitely many different types of infinity. And there is a lovely way to create a whole hierarchy of sets, each set bigger and more infinite than the previous one. And we can explain this by imagining a puzzle.

Write on the top of a page the letters A, B, C, and D. This is a set of 4 letters.

11.7

A B C D

{ } {A} {B} {C} {D}

{AB} {AC} {AD} {BC} {BD} {CD}

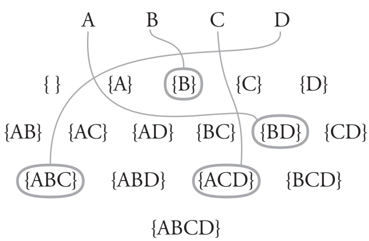
{ABC} {ABD} {ACD} {BCD}

{ABCD}

In the middle of the page, write all possible sets of just 3, or 2, or 1, letter, as well as the empty set (with no letters) and the full set (with 4 letters). There are 16 possibilities.

Without looking, have a friend circle any 4 of the sets and connect each with a leash to 1 of the letters at the top of the page. Your friend will produce a diagram something like what is shown in **figure 11.8**—but you don't see it.

11.8

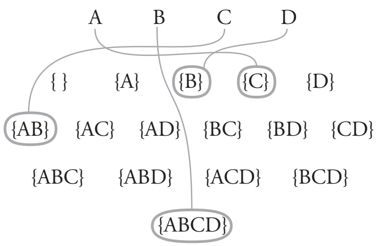


Ask your friend the following questions: Is the letter A sitting in the set to which is it leashed? In this example, the answer is no. A is not in the set {BD}. Is B in the set to which it is leashed? In this example, the answer is yes. Is C in the set to which it is leashed? In this example, the answer is yes. Is D in the set to which it is leashed? In this example, the answer is no.

Now astound your friend by announcing that the set {AD}, the set of letters that gave them no answers, is unleashed. It is still sitting free in the middle of the page—and indeed it is!

The point is that no matter what leash diagram your friend creates, the set of letters that gives the answer no to the question you asked repeatedly—Is this letter in the set to which it is leashed?—will always produce a set that is not linked with any letter.

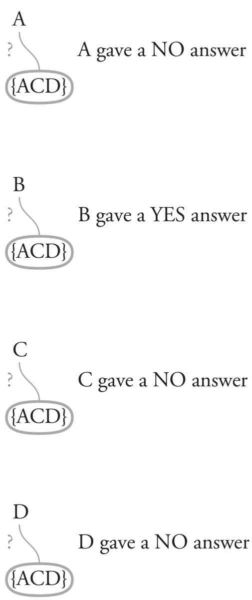
11.9



Another example is shown in **figure 11.9**.

A gives the answer no. B gives the answer yes. C gives the answer no. D gives the answer no.

Indeed, the subset {ACD} is unleashed.



What's the logic behind this? Why must {ACD} be unleashed?

Could {ACD} be leashed to A? Remember that A gave the answer no to the question. There can be no leash between A and {ACD}.

What about B? It gave the answer yes to the question. B couldn't be leashed to {ACD} because B isn't in there.

C and D each also gave the answer no to the question. There can be no leashes to C and D either.

So, the set {ACD} can't be leashed to any letter with a no answer and can't be leashed to any letter with a yes answer. It must indeed be the case that {ACD} is unleashed.

This reasoning works every time, and it proves that in this game there will always be unleashed sets.

The leashing game shows that if you take the set of all sets of a collection of things and attempt to draw leashes, you'll always miss at least 1 set. It is impossible to leash a set with the set of all its sets. In other words, the set of all sets of a set is a bigger set!

This game works for 4 letters, or 5 letters, or 26 letters, or 92 letters (if you want to start making up symbols), and it even works for infinitely many letters: The set of all sets is always bigger than the original set.

We have 1 infinite set: the counting numbers.

Take the set of all sets of counting numbers—the set {1, 7}, the set of all multiples of 3, the set {18, 97, 1013}, all possible sets. This set of sets is bigger. Its size is a bigger infinity.

Do it again: Take the set of all sets of all sets of counting numbers, and you have a bigger infinity still.

Take the set of all sets of all sets of all sets of counting numbers, and you have the next-bigger infinity—and so on forever.

It's difficult to wrap our brains around what a set of sets of sets of sets of sets of sets of sets looks like, but if we keep playing this game with 1 set written out at the top of the page and the set of all sets of it in the middle of the page, leashing will always fail. This means that we will have a bigger infinity sitting in the middle of the page than at the top. And we can keep playing this game over and over again.

We have now proved that there are in fact infinitely many different types of infinity. Cantor shocked the mathematics world when he discovered this.

Cantor worked hard to try to make sense of these different infinities. He did manage to prove that the first infinity after the counting numbers, the set of sets of counting numbers, matches the infinity of the real numbers.

He also showed that the next infinity after this, the set of all sets of sets of counting numbers, matches the set of all possible pictures you can draw on a blank page. The set of all possible pictures is the next size of infinity.

After that, things become difficult to visualize!

Cantor also wondered if there were any infinities sitting between the entries of this list of infinities he constructed. He was never able to answer that question.

Mathematicians working on the logical foundation of mathematics call this question the continuum hypothesis, and they have proved that the current axioms of mathematics are undecided on this matter: One could assume that there are no more types of infinity to be had, and mathematics will be fine with that belief; or one could assume that there are still more infinities out there, and mathematics would still be fine, free of logical inconsistencies.

FURTHER EXPLORATION

READING

Conway and Guy, *The Book of Numbers*.
Tanton, *Thinking Mathematics! Vol. 2*.

PROBLEMS

1. Is "infinity plus 1" bigger than infinity?

11.10

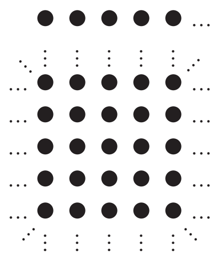
Figure 11.10 is a picture of the infinite set that matches the counting numbers.

Figure 11.11 is that infinite set "plus 1."

11.11

Is it possible to construct a leash pattern between the 2 sets to show that they are the same size, or is the second set genuinely larger than the first?

11.12



2. Demonstrate a leash pattern to show that the 2 infinite sets shown in **figure 11.12** are the same size. (One is a set that matches the counting numbers and the other is a 2-dimensional array of objects extending infinitely far in all directions.)

3. Here is a classic puzzle known as Hilbert's hotel paradox:

> A hotel has an infinite number of rooms numbered 1, 2, 3, 4, …. Currently, the hotel is full—there's a guest in every room—but an extra guest arrives and requests a room. Is it possible to accommodate this 1 extra guest?
>
> The answer is yes, assuming that guests are amenable to changing rooms. Simply have each guest shift 1 room over (from room n to room $n + 1$). This frees up room number 1 for the extra guest and leaves no current guest without accommodation.

a Suppose that 1000 extra guests arrive. Is it possible to accommodate each of these extra guests? How?

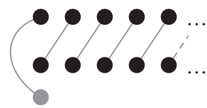
b A busload of tourists next arrives. This bus contains an infinite number of people—numbered 1, 2, 3, 4, …—each looking for a room. Show that it is possible to accommodate this infinite number of people in Hilbert's hotel.

4. Alistair has an infinite number of bags numbered 1, 2, 3, 4, …. Each bag contains an infinite number of marbles numbered 1, 2, 3, 4, …. Is the set of all marbles that Alistair possesses larger than the set of marbles he has in just the first bag?

SOLUTIONS

1. A leashing pattern exists. These 2 sets are the same size. (See **figure 11.13**.)

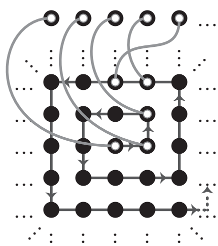
11.13



2. A leashing pattern exists to show that these 2 sets are the same size.

Follow a spiral path in the 2-dimensional array and leash the first element of the spiral to the first element in the top set, the second element in the spiral to the second element in the top set, and so on. (See **figure 11.14**.)

11.14

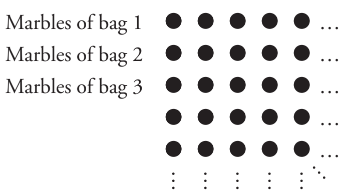


3. **a** Have the guest in room 1 move to room 1001, the guest in room 2 move to room 1002, and so on. This frees up the first 1000 rooms.

b Have the guest in room 1 move to room 2, the guest in room 2 move to room 4, the guest in room 3 move to room 6, and so on. (The guest in room n moves to room $2n$.) This frees up all the odd-numbered rooms for the new guests.

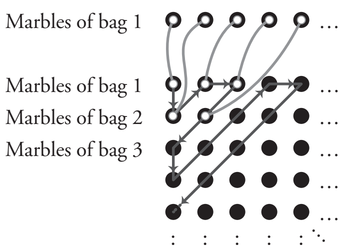
4. We can lay out the marbles of each bag in a row. This gives a diagram of marbles that looks **figure 11.15**.

11.15



Trace a path through this diagram that follows the diagonals. This shows how to match the entire set of marbles with the marbles in bag 1 alone. (Match the first marble in the diagonal path with marble 1, the second marble in the diagonal path with marble 2, and so on, as shown in **figure 11.16**.)

11.16



The total set of marbles that Alistair possesses in all bags is the same size as the set of marbles in his first bag alone.

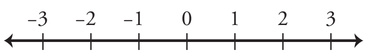
SURPRISE! THE FRACTIONS TAKE UP NO SPACE

LECTURE 12

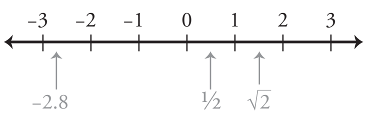
This lecture will establish that, despite seeming to fill up the number line densely, the fractions don't actually fill up all the space on the number line. In fact, this lecture will prove that the fractions take up absolutely no space on the number line. In other words, if you look at a section of the number line that is 3 inches long, none of the length in that segment is due to fractions. Instead, all of the length is produced by the irrational numbers.

FRACTIONS ON THE NUMBER LINE

12.1



12.2

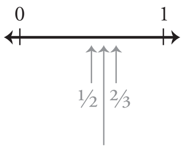


When we think of the set of real numbers, we usually picture a number line.

The number line is infinitely long, stretching infinitely far to the left and infinitely far to the right. The integers are placed on the number line at equally spaced intervals along it.

Every point in the number line represents a number, and every real number is matched with a point on the line. For example, here are the locations of the numbers ½, –2.8, and $\sqrt{2}$ on the number line.

12.3



Certainly, every fraction appears on the number line, too.

But here's the weird thing about fractions: Between any 2 fractions on the number line, no matter how close together they might be, lies another fraction. For example, how might we find a fraction between $\frac{1}{2}$ and $\frac{2}{3}$?

Rewrite the fractions so that they have a common denominator: $\frac{1}{2} = \frac{3}{6}$ and $\frac{2}{3} = \frac{4}{6}$. Can we name a fraction between $\frac{3}{6}$ and $\frac{4}{6}$?

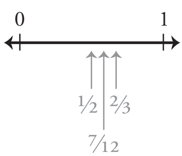
$$\frac{1}{2} = \frac{3}{6} \qquad \frac{2}{3} = \frac{4}{6}$$

$$\frac{3\frac{1}{2}}{6}$$

We can fix this fraction up for the rest of the world by multiplying the numerator and denominator each by 2.

$$\frac{3\frac{1}{2}}{6} = \frac{3\frac{1}{2} \times 2}{6 \times 2} = \frac{7}{12}$$

12.4

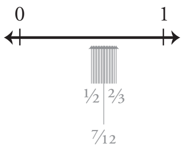


So, $\frac{7}{12}$ is a fraction between $\frac{1}{2}$ and $\frac{2}{3}$.

We can find many, many fractions between $\frac{1}{2}$ and $\frac{2}{3}$. In fact, there are infinitely many fractions sitting between $\frac{1}{2}$ and $\frac{2}{3}$.

And between any 2 of those fractions, we can find infinitely many fractions sitting between them, too—just write the 2 fractions with a common denominator and start listing fractions that sit between them.

12.5



So, it seems that the fractions fill up all the space between ½ and ⅔.

And we can do this for any pair of fractions on the number line—fill up the space between them with infinitely many more fractions.

The fractions take up all the space on the number line. They fill up the whole line completely—so it seems!

But here's the paradox. We know that the fractions can't fill up the entire number line and hit every point on the line: The set of all real numbers is a bigger infinity than the infinity of the fractions. There are many more real numbers than there are fractions. The number line cannot be completely filled with those fractions. Something is out of whack!

LISTING THE FRACTIONS

The geometric series formula, a formula for the following infinitely long sum, is a true statement of algebra:

$$1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1 - x}$$

But if we want to use the formula for actual numbers, then we have to be careful about which values of x we use, because it works for some values of x but not all values.

The formula is valid for all fractions of the form 1 over something, such as $1/10$, $1/100$, and $1/1000$.

$$1/10 + 1/100 + 1/1000 + 1/10{,}000 + \ldots = 1/9$$

$$1/100 + 1/100^2 + 1/100^3 + 1/100^4 + \ldots = 1/99$$

$$1/1000 + 1/1000^2 + 1/1000^3 + 1/1000^4 + \ldots = 1/999$$

12.6

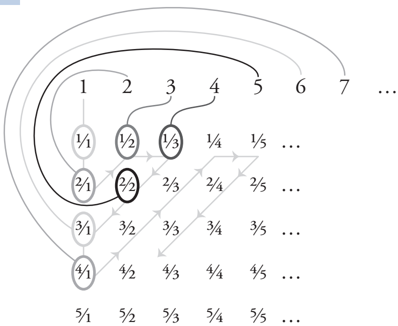


Figure 12.6 is a picture from the previous lecture of how we leashed all the fractions—more specifically, the positive fractions—to the counting numbers.

The fraction $1/1$ (that is, 1) got leashed to the number 1; the fraction $2/1$ (that is, 2) got leashed to the number 2; the fraction $1/2$ got leashed to the number 3; the fraction $1/3$ got leashed to the number 4; and so on.

This means that we are really putting all the fractions in a list: first $1/1$, then $2/1$, then $1/2$, then $1/3$, and so on. If we follow this table and delete any repeat fractions (for example, $2/4$ is the same as $1/2$), then we have the following list of all the fractions:

1, 2, $1/2$, $1/3$, 3, 4, $3/2$, $2/3$, $1/4$, $1/5$, 5, 6, $5/2$, $4/3$, $3/4$, $2/5$, $1/6$, $1/7$, …

These fractions bounce around on the number line as we list them, but the point is that we can list them—all of them. We can list all of the fractions.

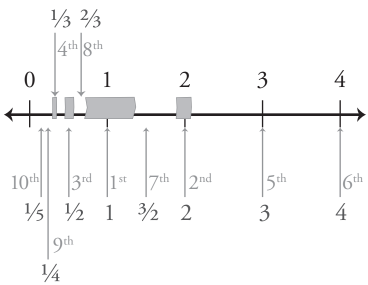
And we can list all of the negative fractions, too, if we want. But let's focus on the positive ones for now.

TAKING UP SPACE

Let's cover the fractions on the number line with ribbon. First, take a piece of ribbon $1/10$ of an inch long and cover the first fraction on our list, which is 1, with that piece of ribbon.

Now cover the second fraction on our list, which is 2, with a piece of ribbon $1/100$ of an inch long.

12.7



Now cover the third fraction on our list, which is ½, with a piece of ribbon $1/1000$ of an inch long.

Also cover the fourth fraction, ⅓, with a piece of ribbon $1/10{,}000$ of an inch long, and so on, covering each fraction on the list with shorter and shorter pieces of ribbon.

If we kept doing this forever, we'd cover every positive fraction on the number line. Even if some of the ribbons overlap, each fraction will still be covered by ribbon. And we could repeat this and cover all of the negative fractions, too.

How much ribbon did we use?

We started with $1/10$ of an inch, and then $1/100$ of an inch, and then $1/1000$ of an inch, and so on. If we add all of these lengths, $1/10 + 1/100 + 1/1000 + \ldots$, we get the answer, $1/9$—that is, we used $1/9$ of an inch of ribbon to cover all of the positive fractions on the number line.

$$1/10 + 1/100 + 1/1000 + 1/10{,}000 + \ldots = 1/9$$

This means that the positive fractions take up no more than $1/9$ of an inch of the entire positive number line. And the negative fractions take up no more than $1/9$ of an inch of space, too.

So, all of the fractions, positive and negative, take up no more than $2/9$ of an inch of space on the number line.

But what if we started with shorter sections of ribbon?

Suppose that we cover the first fraction with $1/100$ of an inch of ribbon, the second with $1/10{,}000$ of an inch of ribbon, the third with $1/1{,}000{,}000$ of an inch of ribbon, and so on, and we use instead the sum $1/100 + 1/100^2 + 1/100^3 + \ldots$, equaling $1/99$—the powers of $1/100$.

$$1/100 + 1/100^2 + 1/100^3 + 1/100^4 + \ldots = 1/99$$

We'd see that, actually, the positive fractions take up no more than $1/99$ of an inch of space on the number line, as do the negative fractions, so the set of all fractions use no more than $2/99$ of an inch of space on the number line—even less!

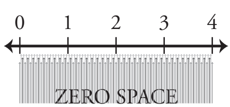
And we can refine this answer further.

If we started with even shorter ribbon and use the powers of 1000 instead—$1/1000 + 1/1000^2 + 1/1000^3 + \ldots$ —we'd conclude that the fractions actually take up no more than $2/999$ of an inch of space on the number line.

$$1/1000 + 1/1000^2 + 1/1000^3 + 1/1000^4 + \ldots = 1/999$$

We can keep doing this and keep concluding that the fractions take up less and less space on the number line. The only conclusion we can draw from this is that the fractions take up 0 space on the number line. We can cover them with less and less and less ribbon.

12.8



The fractions take up no space whatsoever.

So, if you take a section of the number line—for example, 3 inches of it—the fractions contribute 0 inches of length to that segment. All the length in that segment must be coming from the irrational numbers.

People often phrase this conclusion as follows: If you choose a number at random from any section of the number line, for example, by throwing a dart at the number line, the chances of it landing on a fraction is 0.

Zero inches of space out of 3 inches of space gives you 0 chance of landing on a fraction.

This is weird. We humans can only think in terms of fractions.

If you were asked to choose a number on the number line at random, you might choose something like 8.86263. The problem is that you will eventually stop talking, and therefore the decimal you recite will stop. It will be a finite decimal, meaning that your decimal is a fraction.

The only numbers that humans can ever say in human lifetimes are finite decimals, and hence fractions.

Irrational numbers are numbers with infinitely long decimal expansions that go on without pattern. We can never recite one of those. Everything we say must be finite, so everything we think or ever do in mathematics is with fractions. Choose a number at random and we humans can only ever choose a fraction.

We do have some special names for some irrational numbers, such as the square root of 2, but we can never recite their decimal expansions in entirety. We'll never choose the square root of 2 if we have to choose a decimal at random.

Yet the mathematics we just went through shows that we humans are seeing only the merest of slithers of all the possible numbers in the mathematical universe: We see only the fractions, yet they constitute nothing on the number line!

Matters are actually worse than this.

Suppose that we also consider all the numbers we can describe in words, some named irrational numbers like the square root of 2, pi, the cube root of 93, and so on. In fact, let's consider all the numbers we can talk about, all the fractions and irrational numbers that can be described in words.

Now imagine writing out all their names.

> one
> the square root of two
> pi
> pi divided by three
> half
> four and a third
> the second number I thought of today plus one

And so on.

Because these are words, we can put these names in alphabetical order.

> four and a third
> half
> one
> pi
> pi divided by three
> the second number I thought of today plus one
> the square root of two

So, in principle, every number we can possibly describe—all the integers, all the fractions, a whole slew of irrational numbers—can be put in a list, an alphabetical list. And we can look at the first number on the list and cover that number on the number line with a piece of ribbon, look at the second number on the list and cover that number on the number line with a shorter piece of ribbon, and so on, and go through the whole process we conducted previously to conclude that all of the numbers on our list take up 0 space on the number line.

In other words, we have just proved that the set of all numbers we can possibly describe in words also take up absolutely no space on the number line!

This means that the bulk of the number line is taken up with numbers we can't even describe in words.

But it gets even worse.

We often like to think that numbers, if not described by words, can be described by equations. But we can also write out equations in words, using the letter x for the variable. This means that even equations can be described by letters of the alphabet and therefore included in our list of numbers put in alphabetical order by their descriptions. So, we must conclude that the majority of numbers on the number line cannot be described in words nor be described by mathematical equations. The majority of numbers on the number line are just truly inaccessible: We'll never do mathematics with them, we'll never know them, and we'll never experience them!

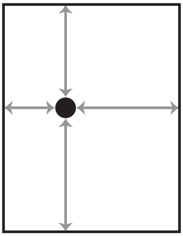
FURTHER EXPLORATION

WEB

Tanton, "Fractions Take Up No Space."
http://www.jamestanton.com/?p=848
———, "The Geometric Series Formula."
http://www.jamestanton.com/?p=723

PROBLEMS

12.9



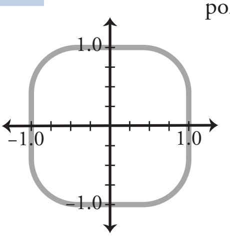
1. List 99 different fractions that lie between ⅛ and ⅑.

2. A point is chosen at random on a piece of 8 ½-by-11-inch sheet of paper. What are the chances that each of the 4 distances of that point from the sides of the paper is a fractional number of inches?

3. How can we show that $0.4999\ldots = 0.4\overline{9}$ is the same as 0.5?

4. It is known that there are no nonzero whole-number solutions to the equation $a^4 + b^4 = c^4$. Use this fact to explain why the curve $x^4 + y^4 = 1$ drawn in the plane avoids passing through points with both coordinates that are nonzero fractions.

12.10



SOLUTIONS

1. The fractions $\frac{1}{8.01} = \frac{100}{801}$, $\frac{1}{8.02} = \frac{100}{802}$, $\frac{1}{8.03} = \frac{100}{803}$, … $\frac{1}{8.99} = \frac{100}{899}$, for example, do the trick.

2. Looking at just the distance of the point from the left edge, the chances that a value between 0 and 8.5 chosen at random is a fraction is 0. Similarly, the chances that a value between 0 and 11 chosen at random is a fraction is also 0. The probability that we choose a point with the properties we seek is 0.

3. Let $G = 0.4999\ldots$. Then,

   $$10G = 4.999\ldots$$
   $$100G = 49.999\ldots$$

   Subtract these to see that $90G = 45$, forcing $G = \frac{1}{2}$.

4. Suppose that the curve $x^4 + y^4 = 1$ passes through the point $(r/s, p/q)$ with fractional coordinates, neither equal to 0. Then, $(r/s)^4 + (p/q)^4 = 1$. This can be rewritten as $r^4/s^4 + p^4/q^4 = 1$.

   Multiplying through by s^4 and q^4 gives $r^4q^4 + p^4s^4 = s^4q^4$, which is $(rp)^4 + (ps)^4 = (sq)^4$. This is a solution to the equation $a^4 + b^4 = c^4$. But there are no whole-number solutions to this equation. Our initial assumption that the curve passes through a point with fractional coordinates must be wrong.

VISUALIZING PROBABILITY

LECTURE 13

Mankind has been thinking about chance for millennia. At the very least, gambling games have been in existence for centuries, and scholars have wondered about, analyzed, and computed likelihoods of events for those games. People have been aware for a long time, for example, that if you toss a fair coin a large number of times, that coin will land on a particular face about half of those times. This lecture will explore the mathematics of probability.

PROBABILISTIC EVENTS

What do we like to believe about probabilistic events? We certainly believe that for many basic actions, such as flipping a coin or rolling a die, that each possible outcome of the action has a certain inherent number associated with it, which we call its probability.

For example, the number we associate with the result of getting a head when flipping a fair coin is 50%—that is, ½. The number we associate with the result of seeing a 6 when rolling a die is ⅙.

Where do these numbers come from? We like to believe that they come from our experience of things—that they come from the data we see.

In flipping a coin a large number of times, we see a head about 50% of those times. We associate the number ½, then, with the outcome of tossing a head. If we roll a pair of dice a large number of times, data shows that a pair of 1s appears about 1 in 36 times. We associate the number 1⁄36 with the event of rolling a pair of 1s.

So, in general, in associating a number p% with a certain outcome of an experiment, we are saying that if we perform that experiment many times, then we'd expect to see that outcome appear about p% of the time.

But we know that things aren't perfect in the world and that we are likely not to see exactly p% of the outcomes we expect. For example, it you toss a coin 10 times, you probably won't see exactly 5 heads and exactly 5 tails. But we do believe that the more times you repeat the action, the closer the proportion of outcomes will be to the value 50%. Toss a coin a million times, and the proportion of heads you see will most likely be really close to ½.

This thinking works in reverse. Suppose that you toss a coin 500 times and it lands heads 407 of those times. Then you would likely conclude that the coin is biased—biased in a way that it has about an 80% chance of landing heads.

This intuitive idea to probability theory is quite powerful. We can already answer some basic textbook questions just by thinking our way through them with this one idea in mind. For example, here's a classic textbook question:

A person will toss a coin and then roll a die.

What are the chances that the person will see a head followed by a 1 or a 6?

13.1

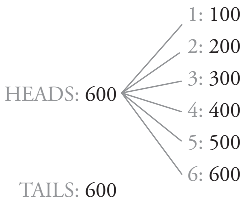
To answer this question, imagine running this double-barreled experiment a large number of times. Let's toss a coin and then roll a die, for example, 1200 times.

HEADS: 600

For the coin tosses, we expect to see a heads about half the time and tails about half the time—that is, in the ideal situation, we expect to see 600 heads and 600 tails.

TAILS: 600

13.2

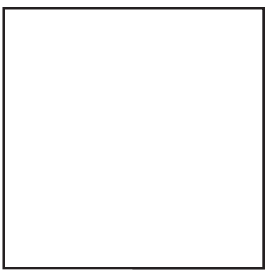


For those 600 times we see a head, we will also be rolling a die. About ⅙ of those times we expect to roll a 1, about ⅙ of the time we expect to roll a 2, and so on. In other words, of these 600 rolls, we expect each number to appear 100 times in the ideal situation.

How many times do we see a head followed by a 1? 100 times.

How many times do we see a head followed by a 6? 100 times.

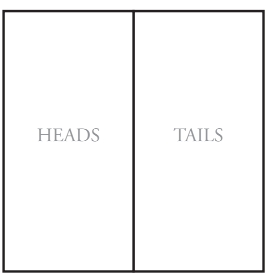
13.3



So, out of 1200 runs of our experiment, we see what we want 200 times—that is, 200 out of 1200 runs of the experiment give us what we want. This mental data suggests that our chances of seeing a head followed by a 1 or a 6 is $\frac{200}{1200}$, or ⅙.

But it is much easier to think through this visually. Instead of choosing a particular count of experiments to work with—we chose the number 1200—let's not specify a number and just draw a rectangle to represent a large number of experiments.

13.4



In tossing a coin, we expect half the outcomes to be heads and half to be tails.

With the roll of a die, we expect, among the heads, each particular number on the die to appear about ⅙ of the time.

13.5

HEAD and 1	TAIL and 1
HEAD and 2	TAIL and 2
HEAD and 3	TAIL and 3
HEAD and 4	TAIL and 4
HEAD and 5	TAIL and 5
HEAD and 6	TAIL and 6

We see the entire rectangle of outcomes and the visual proportion of times we expect to get a head followed by a 1 or a 6. It is $2/12$—that is, $1/6$—of the rectangle.

If you like, we can put in the full details of the situation. After all, we do roll a die each time we toss a tail, too. Here's the rectangle of outcomes in full detail. It is a little easier now to see that the proportion of outcomes we are interested in really does correspond to $2/12$ of the entire rectangle.

Thinking visually in probability problems is wonderful. Each action, such as the toss of a coin or the roll of the die, is like being presented with a fork in the road—the outcome determines which path you are next to follow and therefore which part of a diagrammatic rectangle you land in.

THE PROBLEM OF POINTS

In 1654, a French nobleman named Antoine Gombaud, chevalier de Méré, wrote to the most prominent French mathematician of the time, Blaise Pascal, asking for advice on the general "problem of points." He was wondering what to do in a situation like the following:

Two friends—for example, Albert and Bilbert—each lay down \$100 in a friendly best-of-7 tennis game. But rain interrupts play after just 4 games, with 1 person—let's say Albert—having won 3 games and the other, Bilbert, just 1. How should the \$200 pot be divided between the 2 players to properly reflect the likelihood of each winning?

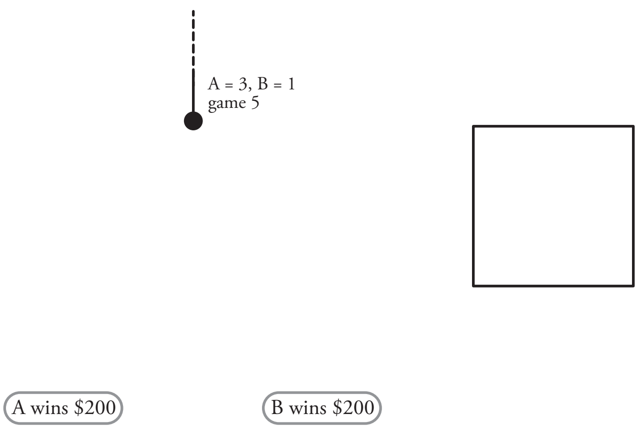
Pascal shared this problem with his colleague Pierre de Fermat, and both gentlemen developed mathematical ideas to analyze this problem, and others like it. Probability theory, as we recognize it today, was born.

We'll come up with a possible solution to this problem, but we do have to make an assumption: that Albert and Bilbert are equally strong players—that they each have a 50% chance of winning any particular game. We can, of course, challenge this assumption, but as a start to answering the question, let's go with it for now.

Let's start by drawing a diagram for what could happen if they were able to continue the games. We'll also draw the associated rectangle as we go along.

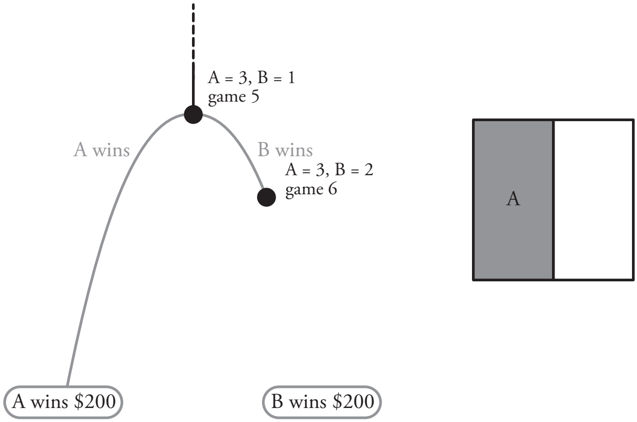
Albert has 3 wins, Bilbert has 1, and the path starts by approaching the play of a fifth game.

13.6



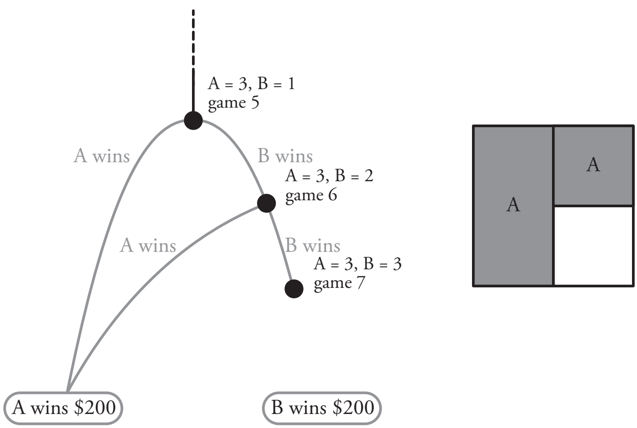
The fifth game represents a fork in the diagram. If Albert wins it, then he has 4 games under his belt and therefore wins the best of 7 and the $200 pot. If Bilbert wins it, then Albert has a score of 3, Bilbert has a score of 2, and they move on to a sixth game.

13.7



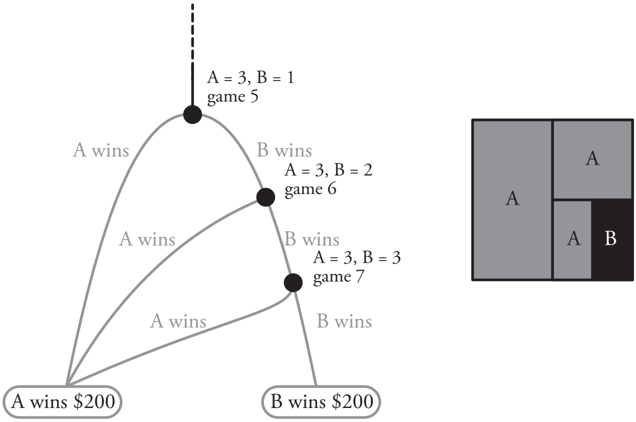
The sixth game represents another fork. If Albert wins, then he wins best of 7 and gets the $200 pot. If, on the other hand, Bilbert wins, then they both have a score of 3, and they move on to a seventh game.

13.8



We then have another fork in the road. If Albert wins the seventh game, he wins the pot. If Bilbert wins, then Bilbert gets the $200.

13.9

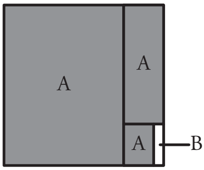


We see from the rectangle that if the gentlemen would keep playing games 5, 6, and 7, Bilbert wins the pot $\frac{1}{8}$ of the time while Albert wins $\frac{7}{8}$ of the time. This suggests, following these likelihoods, that the men should split the pot in a 7:1 ratio—that is, divide the pot of $200 into 8 equal parts ($25) and give Albert 7 of those parts ($175) and Bilbert 1 of those parts ($25).

That's one way to divide the pot in a manner that arguably reflects likelihoods of wining.

But we made a beginning assumption that each gentleman is equally likely to win any individual game. One might argue that evidence suggests otherwise. Before the rain, Albert won 3 games and Bilbert won only 1. So, we might argue that Albert has a $\frac{3}{4}$ chance of winning any individual game, according to the evidence.

13.10



In which case, we should adjust our picture. Rather than split regions into half as we go along, we should split them in a way that gives Albert ¾ of each piece of the region being split. We can get the picture for this just by sliding each of our lines a tad.

One can check that this modified rectangle suggests that Bilbert wins overall only 1⁄64 of the time and Albert wins 63⁄64 of the time. This would mean that we split the pot in a 63:1 ratio, giving Albert $196.875 and Bilbert $3.125.

But Bilbert, of course, might argue that this isn't fair—that he is actually the stronger player and was just laying low the first few games to give Albert a false sense of confidence! Alas, we'll never escape human psychology and human emotions.

But no matter the beginning assumptions we choose to make, we now have the mathematical means to compute concrete answers in response to those assumptions.

FURTHER EXPLORATION

WEB

Garsia and Zabrocki, "Let's Make a Deal." (Explore the Monty Hall problem.)
http://math.ucsd.edu/~crypto/Monty/monty.html

Tanton, "The Astounding Power of Area."
http://gdaymath.com/courses/astounding-power-of-area/

———, "Curriculum Essay: June 2015."
http://www.jamestanton.com/wp-content/uploads/2012/03/Curriculum-Essay_July-2015_Basic-Probability.pdf

———, "The Infamous Boy-Boy Paradox."
http://gdaymath.com/lessons/powerarea/4-6-more-practice/

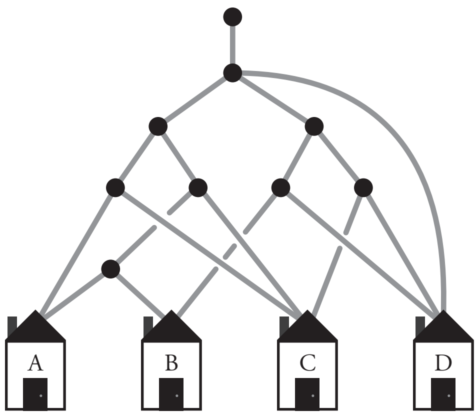
READING

Devlin, *The Unfinished Game.*

PROBLEMS

1. A large number of people walk down the following complex of garden paths with equal counts of people heading down each choice of path at any fork. What fraction of people end up in each of the 4 houses?

13.11

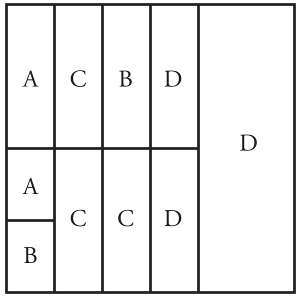


2. A bag contains 2 cards. The first card is blue on one side and white on the other. The second card is blue on both sides. You reach into the bag and slide out a card to see that it has a blue face. What are the chances that you have pulled out the blue/blue card?

SOLUTIONS

1. **Figure 13.12** shows the matching rectangle diagram for the garden paths.

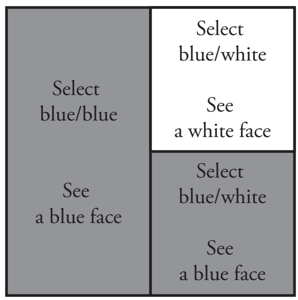
13.12



We see that equal counts of people end up in houses A and B—namely, $\frac{1}{12} + \frac{1}{24} = \frac{1}{8}$ of them—and $\frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$ of people end up in house C, leaving $\frac{1}{2}$ of people for house D (and this does indeed match $\frac{1}{12} + \frac{1}{12} + \frac{1}{3}$).

2. If this exercise is repeated a large number of times, the blue/blue card will be selected half the time, and the blue/white card will be selected half the time. In the first case, we are sure to see a blue face. In the second case, we will see a blue face half of those times. **Figure 13.13** depicts this information.

13.13



In the scenario of the question, we know that we are located in a shaded region of the rectangle. What are the chances we're the shaded portion corresponding to the blue/blue card? The answer is $\frac{2}{3}$.

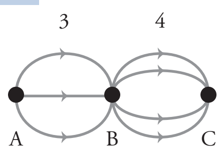
VISUALIZING COMBINATORICS: ART OF COUNTING

LECTURE 14

A coin has only 2 faces, so the chances of tossing a head are 1 in 2. But hands in a card game offer many options and possibilities—and counting how many possibilities there are can be very tricky. Problems like this come from a branch of mathematics called combinatorics, the art of counting complicated things. When you first study complicated counting problems, it seems that each problem has a special technique for answering it, each requiring a special formula. But, as you will learn in this lecture, all of them are actually the same problem in disguise, and there is only one technique needed to solve them all.

THE MULTIPLICATION PRINCIPLE

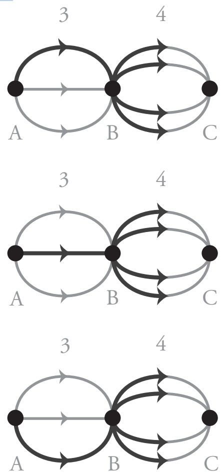
14.1



In this picture, there are 3 1-way paths that take you from point A to point B, and there are 4 paths that get you from point B to point C. How many different routes are there to go from A to C?

A quick response might be to say 7, for 3 + 4—3 choices followed by 4 choices. But the answer is actually 12, from 3 × 4.

14.2



4 + 4 + 4 options

We certainly have 3 options for getting from A to B. If we take the top, we are then presented with 4 options for next getting from B to C.

Taking the middle route from A to B also presents us next with 4 options for getting from B to C, as does taking the bottom route. We thus have 4 + 4 + 4 options in all—3 groups of 4. That's multiplication: # choices from A to B, followed by 4 choices from B to C, makes for 3 × 4, or 12, options overall.

The multiplication principle states that if there are *a* ways to complete a first task and *b* ways to complete a second task, and no choice of outcome in any one task in any way affects the choice of the outcome in the other, then there are *a* × *b* ways to complete both tasks together.

COUNTING WORDS

How many ways are there to rearrange the letters in the name Jim?

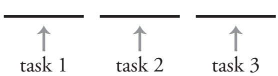
14.3

JIM

JIM	IJM	MIJ
JMI	IMJ	MJI

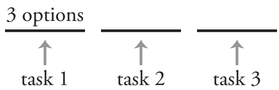
This problem is straightforward enough that we can just list all the ways—6 ways in total.

14.4



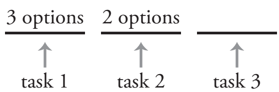
But let's answer this question using the multiplication principle. We need to use the letters J, I, and M to make a 3-letter word. This means that we have 3 slots to fill, each with a letter. Therefore, we have 3 tasks at hand: Fill in the first slot, then fill in the second slot, then fill in the final slot.

14.5

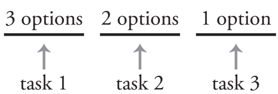


How many ways are there to complete task 1, filling in the first slot? We have 3 letters to choose from—J, I, and M—so there are 3 options for putting a letter in slot 1.

Once we've completed task 1, we have 2 letters left to work with. This means that there will be 2 options for completing task 2: putting a letter in the middle slot.



We've used 2 letters now, meaning that no matter what choices we've made along the way, there will only be 1 option for which letter to put in the third slot: the 1 remaining letter.



So, there are 3 tasks: 3 ways, 2 ways, 1 way. By the multiplication principle, there are $3 \times 2 \times 1$, which equals 6, ways to complete all 3 tasks together—that is, there are 6 ways to arrange the letters J-I-M.

Products of integers like these—$3 \times 2 \times 1$—come up a lot in these types of counting problems. Mathematicians have a special name and special notation for these products. They are called factorials, and they use an exclamation point to denote them.

$1! = 1$
$2! = 2 \times 1 = 2$
$3! = 3 \times 2 \times 1 = 6$
$4! = 4 \times 3 \times 2 \times 1 = 24$
$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$
$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$
$9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362{,}880$
$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800$

For positive integer *n*, *n* factorial (*n*!) is the product of all the integers between 1 and *n*. Factorial numbers grow very large very quickly.

There were 3 × 2 × 1 = 3! ways to arrange the 3 letters of Jim. If we had a name with 8 distinct letters, then there would be 8! ways to arrange the letters of that name. In general, there are *n*! ways to arrange *n* distinct letters.

14.6

BOB

BOB BBO OBB

How many ways are there to rearrange the letters in the name Bob, with a repeat B? How could we handle repeat letters? By brute force, we can see that there are just 3 ways to rearrange Bob.

But how would we answer this question using our counting techniques?

To deal with the repeat Bs, we could make the 2 Bs look different by adding subscripts to them.

14.7

$\mathbf{B_1OB_2}$

$$\begin{array}{ccc} B_1OB_2 & B_1B_2O & OB_1B_2 \\ B_2OB_1 & B_2B_1O & OB_2B_1 \\ \downarrow & \downarrow & \downarrow \\ \text{BOB} & \text{BBO} & \text{OBB} \end{array}$$

Now we have 3 distinguishable letters, so the problem is just like the previous problem involving the name Jim: There are 3! ways to rearrange 3 distinct letters.

But those subscripts were an invention, and we have to erase them. But if we do erase them, we see that all of our answers collapse in pairs.

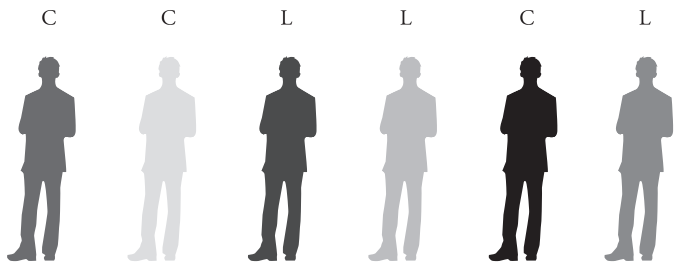
So, our answer of 3! was off by a factor of 2. The answer is actually 3! divided by 2, which is 6 divided by 2, which is 3. There are 3 ways to arrange the letters of Bob.

PROBLEM 1

How many ways can we select 3 people for a committee from a group of 6 people?

This picture show that this is just a word problem. We must count the number of ways to make a 6-letter word with three Cs, for "chosen," and three Ls, for "lucky"—no one really wants to be on a committee.

14.8



We know that there are 6!/(3!3!) ways to make 6-letter words: 1 3! in the denominator for 3 Cs and another 3! in the denominator for 3 Ls. This works out to 20 possible ways to make a committee of 3 from a group of 6.

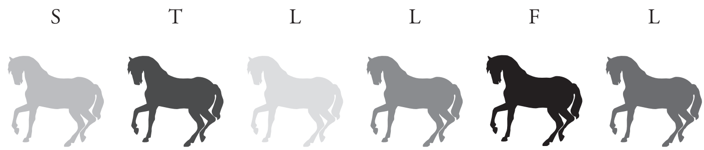
$$\frac{6!}{3!3!} = \frac{720}{6 \times 6} = 20$$

PROBLEM 2

Six horses run a horse race. Assuming that there are no ties, how many different options are there for first, second, and third place?

We want to make 6-letter words using 1 F, 1 S, 1 T, and 3 Ls ("first," "second," "third," and 3 "losers").

14.9



There are 6!/(1!1!1!3!) ways, which works out to 120 ways. Thus, there are 120 different possible outcomes to this horse race.

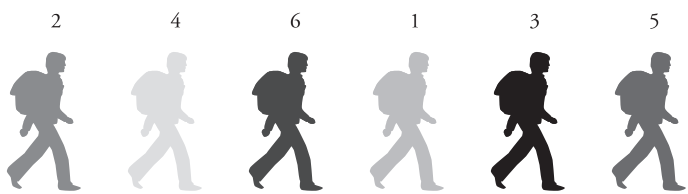
$$\frac{6!}{1!1!1!3!} = \frac{720}{1 \times 1 \times 1 \times 6} = 120$$

PROBLEM 3

In how many different ways can we arrange 6 students in a line from front to back?

This time, we want to arrange 6 different symbols. We can use the symbols 1, 2, 3, 4, 5, and 6 for first in line, second in line, and so on.

14.10



There are 6!—that is, 720 ways—to arrange 6 distinct symbols. There are thus 720 different ways to put 6 students in a line.

Because we have one 1, one 2, one 3, one 4, one 5, and one 6, we could write the answer as 6!/(1!1!1!1!1!1!). This denominator takes into account each symbol in the word. Even though this seems inefficient, it is still correct thinking and gives the same answer.

$$\frac{6!}{1!1!1!1!1!1!} = \frac{720}{1 \times 1 \times 1 \times 1 \times 1 \times 1} = 720$$

PROBLEM 4

One hundred third graders take part in a feel-good running race. The first 50 to cross the finish line will all be called "winners." The next 20 to cross the line will be called "as good as winners." The last 30 to cross the line will be called "may as well have been winners." How many different outcomes are possible for this race?

Let's visualize 100 children in a row, instead of drawing a picture.

In running this race, 50 will be labeled W for "winner." Another 20 will be labeled A for "as good as winners," and the remaining 30 will be labelled M for "may as well have been winners." So, with all the children lined up in a row, the result of the race will look like a word 100 letters long composed of 50 Ws, 20 As, and 30 Ms. How many possible such words are there? 100!/(50!20!30!).

$$\frac{100!}{50!20!30!}$$

The value of this is a very large number; it's 43 digits long.

PROBLEM 5

In an office of 20 people, the boss is going to select 6 people at random for a committee. What are the chances that Albert and Bilbert both end up on that committee?

Let's first figure out how many 6-person committees there are in total. In your mind, line up the 20 office people in a row. We must label 6 of these people as C, for "chosen," and 14 as L, for "lucky." So, we are counting 20-letter words composed of 6 Cs and 14 Ls. There are 20!/(6!14!) of those words. This works out to 38,760 possible committees.

$$\frac{20!}{6!14!} = 38{,}760$$

How many of those committees have both Albert and Bilbert on them?

To make a committee with those 2 men on it, start by assigning Albert and Bilbert to the committee. Then, line up the remaining 18 people in a row. Label 4 of these people as C for "also chosen to be on with Albert and Bilbert" and 14 as L for "lucky."

Now we're counting 18-letter words with 4 Cs—we already have Albert and Bilbert—and 14 Ls. There are 18!/(4!6!) of those. That works out to be 3060. So, there are 3060 committees that have both Albert and Bilbert on it.

$$\frac{18!}{4!14!} = 3060$$

So, 3060 out of 38,760 committees have both men on the committee. That proportion—3060 divided by 38,760—is about 0.08. This means that there is about an 8% chance that a randomly formed committee will have both men on it.

FURTHER EXPLORATION

WEB

Tanton, "Permutations and Combinations."
http://gdaymath.com/courses/permutations-and-combinations/

READING

Tanton, *Mathematics Galore!* (More on counting lines, intersections, and regions in a circle.)

PROBLEMS

1. The word "abstemiously" is one of the few words in the English language in which each vowel (including y) appears exactly once and in correct order.

a In how many ways can the letters of "abstemiously" be arranged?

b In how many ways can the letters of "abstemiously" be arranged while keeping the vowels in their current positions?

2. There are 15 people in an office, and 2 committees must be formed. The first committee will consist of a group of 6, with 1 person named chair and 2 people named co-secretaries for that committee. The second committee will consist of 9 people, with 3 named cochairs, 1 secretary, and 2 co-treasurers. Assuming that no person can be on both committees and no person can assume more than 1 role within a committee, how many different ways are there to form these 2 committees?

(Note: Questions like this give school mathematics a bad name! However, with our newfound method, even absurd questions like this become manageable.)

3. From 10 people, 4 are to be chosen for a team. How many ways are there to select the team if:

a the order in which team members are chosen is not important?

b the order in which team members are chosen is considered important?

SOLUTIONS

1. **a** There are 12!/(2!1!1!1!1!1!1!1!1!1!1!) = 479,001,600/2 = 239,600,800 ways to arrange the 12 letters with 2 repeated s's.

b Our job is to arrange 6 letters (b, s, t, m, s, l) in the following slots: a_ _ _ e _ iou _ _ y. There are 6!/(2!1!1!1!1!) = 720/2 = 360 ways to do this.

2. We must assign the following labels:

1 person is "chair of committee 1."
2 people are "co-secretaries of committee 1."
3 people are "ordinary members of committee 1."
3 people are "cochairs of committee 2."
1 person is "secretary of committee 2."
2 people are "co-treasurers of committee 2."
3 people are "ordinary members of committee 2."

Lining up these people in a row and assigning labels makes the puzzle a word-counting problem. There are 15!/(1!2!3!3!1!2!3!) = 1,513,512,000 possibilities.

3. **a** Line the 10 people in a row. We must assign 4 people the label "chosen" and 6 people the label "not chosen." This makes the question a word-counting problem. There are 10!/(4!6!) = 210 possibilities.

b Line the 10 people in a row. We must assign 1 person the label "chosen first," 1 the label "chosen second," 1 the label "chosen third," 1 the label "chosen fourth," and 6 people the label "not chosen." This makes the question a word-counting problem. There are $10!/(1!1!1!1!6!) = 5040$ possibilities.

Note: School mathematics call the scenario of part a) a combination problem and the scenario of part b) a permutation problem. With the word-counting approach, there is no need to distinguish the 2.

VISUALIZING PASCAL'S TRIANGLE

LECTURE 15

Pascal's triangle is a very famous array of numbers in mathematics that is named, in the Western world, after 17th-century French mathematician Blaise Pascal, who used the triangle extensively in his work on probability theory. The triangle has been known and studied for many centuries before Pascal's time in Asia, India, and Persia. In this lecture, you will learn about Pascal's triangle, as well as some interesting patterns that can be discovered within it.

PASCAL'S TRIANGLE

Walking only rightward or downward steps, how many different paths are there from the top-left corner of a grid of squares to each particular cell on the grid?

15.1

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	
1	3	6	10	15	21	28		
1	4	10	20	35	56			
1	5	15	35	70				
1	6	21	56					
1	7	28						
1	8							
1								

15.2

R R R R R

D	1	1	1	1	1	1	1	1	1
D	1	2	3	4	5	6	7	8	
D	1	3	6	10	15	21	28		
D	1	4	10	20	35	56			
	1	5	15	35	70				
	1	6	21	56					
	1	7	28						
	1	8							
	1								

$$\frac{7!}{3!4!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} = 35$$

We can develop formulas for each number in this array. For example, the number 35 comes from walking 3 steps to the right and 4 steps down in some order. There are 7!/(3!4!) ways to arrange 3 Rs and 4 Ds, which works out to be 35.

When we tilt this picture of numbers 45°, we get Pascal's triangle, the version of the array with which most people are familiar.

15.3

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮ 56

In the triangle, each entry in the array is the sum of the 2 entries just above it. For example, 5 + 10, 4 + 1, and 35 + 21 will give a number on the next row—15, 5, and 56, respectively.

This is how people generally learn to construct the triangle when they first see it. (See **figure 15.3**.)

15.4

0th row 1
1st row 1 1
2nd row 1 2 1
3rd row 1 3 3 1
⋮ 1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
7th row 1 7 21 35 35 21 7 1
0 1 2 3 4 3 2 1 0
⋮

$$\frac{7!}{3!4!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} = 35$$

Let's call the top row of the triangle the zeroth row, the next row the first row, and so on.

Look at entry 35, for example. It is in the seventh row and is the entry 0, 1, 2, 3 places in from the left and 0, 1, 2, 3, 4 places in from the right. The formula for this number is 7!/(3!4!).

In general, to find the value of the entry on the n^{th} row of the triangle (starting the count at 0), count how many places that entry lies in from the left—suppose that is a places—and how many places that entry lies in from the right—suppose that is b places—and the value of that entry is $n!/(a!b!)$.

PATTERNS IN PASCAL'S TRIANGLE

There are many beautiful patterns to be discovered in Pascal's triangle. You may know some of these patterns already. For example, add the entries in each row of the triangle. Is the sum of the entries in any single row always double the sum of the entries of the previous row?

15.5

1 = 1
1 + 1 = 2
1 + 2 + 1 = 4
1 + 3 + 3 + 1 = 8
1 + 4 + 6 + 4 + 1 = 16
1 + 5 + 10 + 10 + 5 + 1 = 32
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

Let's think about the next sum 1 + 6 + 15 + 20 + 15 + 6 + 1. Of course, we can check that the answer is 64, but we want to know philosophically why it has to have an answer that is double the previous sum. Recall that each entry in the triangle is the sum of the 2 entries just above it. This is even true for the 1s at the beginning and end of each row—each 1 is the sum of the blank and the 1 above it.

15.6

1 = 1
1 + 1 = 2
1 + 2 + 1 = 4
1 + 3 + 3 + 1 = 8
1 + 4 + 6 + 4 + 1 = 16
1 + 5 + 10 + 10 + 5 + 1 = 32
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

Now look at the sum 1 + 6 + 15 + 20 + 15 + 6 + 1. The 1 is the sum of the blank and the 1 above it.

The 6 is really the sum of the 1 and the 5 just above it. See the 1 being double counted.

15.7

1 = 1
1 + 1 = 2
1 + 2 + 1 = 4
1 + 3 + 3 + 1 = 8
1 + 4 + 6 + 4 + 1 = 16
1 + 5 + 10 + 10 + 5 + 1 = 32
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

15.8

1 = 1
1 + 1 = 2
1 + 2 + 1 = 4
1 + 3 + 3 + 1 = 8
1 + 4 + 6 + 4 + 1 = 16
1 + 5 + 10 + 10 + 5 + 1 = 32
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

The 15 is the sum of 5 and the 10 above it. Now the 5 is being double counted.

The 20 is the 10 and the 10 above it. The 15 is the 10 and the 5 above it. The 6 is the 5 and the 1 above it, and the final 1 is the 1 and the blank above it.

We see now that adding 1 and 6 and 15 and 20 and 15 and 6 and 1 is just adding the entries of the previous row, doubled! The answer must indeed be double the previous answer. The previous answer was 32, so the sum of the entries in this sixth row must be 64.

We've just discovered that the entries in any row of Pascal's triangle sum to a power of 2, just by chasing through a visual pattern.

15.9

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

1 + 3 + 6 + 10 = 20

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

1 + 5 + 15 = 21

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

1 + 1 + 1 + 1 + 1 + 1 = 6

Here's another curious pattern. Choose any 1 on the side of Pascal's triangle, come down a diagonal for a while, and then turn 90° downward 1 more step to complete a picture of a sock or a stocking. Then, the numbers in the leg of the stocking always sum to the number in the toe. For example, 1 + 3 + 6 + 10 is 20, and 1 + 5 + 15 is 21, and a sum of six 1s is 6.

15.10

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
1   7  21  35  35  21  7   1
            ⋮
```

20 = 10 + 10

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
1   7  21  35  35  21  7   1
            ⋮
```

20 = 4 + 6 + 10

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
1   7  21  35  35  21  7   1
            ⋮
```

20 = 1 + 3 + 6 + 10

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
1   7  21  35  35  21  7   1
            ⋮
```

20 = 1 + 3 + 6 + 10

We can explain this pattern visually, too. Consider the 20 on the sixth row, for example. That 20 is really the sum of the two 10s just above it: 20 is 10 + 10.

But the left 10 is really the sum of the 4 and 6 above it: 20 is 4 + 6 + 10.

But the 4 is really the sum of the 1 and 3 above it: 20 is 1 + 3 + 6 + 10.

And the 1 is the sum of the blank and 1 above it. In the end, we see that the 20 is the sum of the numbers in the leg of its stocking.

Here's something else curious. Have you ever noticed that the powers of 11 are linked to Pascal's triangle? 11 is just 11. 11 squared is 11 × 11 = 121. 11 cubed is 11 × 11 × 11 = 1331. 11 to the fourth power is 11 × 11 × 11 × 11 = 14,641.

$11^1 = 11$

$11^2 = 121$

$11^3 = 1331$

$11^4 = 14{,}641$

People say that a number to the zeroth power is 1, and this fits with Pascal's triangle, too.

$11^0 = 1$

$11^1 = 11$

$11^2 = 121$

$11^3 = 1331$

$11^4 = 14{,}641$

Matters seem to break down for 11 to the fifth power. On a calculator, we get $11^5 = 161051$. But if we write this in a dots-and-boxes way, this is the same as 1, 5, 10, 10, 5, 1—1 unit, 5 tens, 10 tens, 10 hundreds, 5 thousands, and 1 ten thousand. It does actually match the fifth row of Pascal's triangle after all!

$11^0 = 1$
$11^1 = 11$
$11^2 = 121$
$11^3 = 1331$
$11^4 = 14{,}641$
$11^5 = 161{,}051 = 1 \mid 5 \mid 10 \mid 10 \mid 5 \mid 1$

So, why is there this strange connection to the powers of 11?

The answer can be found in algebra class, in which you might have had to memorize the following equations:

$(x + y) = x + y$
$(x + y)^2 = x^2 + 2xy + y^2$
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

We can see the entries of Pascal's triangle in this algebra—especially if we write in the coefficients of 1, as well:

$(x + y) = 1x + 1y$
$(x + y)^2 = 1x^2 + 2xy + 1y^2$
$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$
$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Why is there this connection to the triangle in algebra class?

Let's look at $(x + y)$ to the fourth power, for example:

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y)$$

How do you expand parentheses? You select 1 term from each set of parentheses, multiply them together, and add all possible combinations. For example, if we select an x from each set of parentheses, we'll get the term x times x times x times x in the expansion—that is, x^4.

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y) = x^4 + \ldots$$

If we select an x, then a y, then a y, and then an x, we get the term x times y times y times x, which in algebra can be rewritten x^2y^2.

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y) = x^4 + x^4y^2 + \ldots$$

In fact, we'll get another term x^2y^2 from selecting x, then x, then y, then y, and another from selecting y, then x, then y, then x, and another form selecting y, then y, then x, then x—and so on. In fact, for every way that we can arrange 2 x's and 2 y's, we get the term x^2y^2 in the expansion of $(x + y)^4$.

$$(x + y)^4 = x^4 + x^2y^2 + \ldots$$

From: *xxyy*
yxxy
yyxx $\quad \dfrac{4!}{2!2!} = 6$ ways
yxyx
xyxy
xyyx

And we know how many ways there are to arrange 2 x's and 2 y's. There are 4!/(2!2!) ways. This is a formula in Pascal's triangle. The number 4!/(2!2!), which equals 6, is the middle number of the fourth row of the triangle. The number of times the expression x^2y^2 appears in expanding $(x + y)^4$ is 4!/(2!2!) times.

$$(x+y)^4 = x^4 + \underset{\substack{\uparrow \\ \frac{4!}{2!2!}}}{6x^2y^2} + \dots$$

The expression x^3y comes from 3 x's and 1 y, and we know that there are 4!/(3!1!), which equals 4, ways to make that happen—again an entry from the fourth row of Pascal's triangle.

$$(x+y)^4 = x^4 + \underset{\substack{\uparrow \\ \frac{4!}{3!1!}}}{4x^3y} + \underset{\substack{\uparrow \\ \frac{4!}{2!2!}}}{6x^2y^2} + \dots$$

In this way, we can see that all the numbers that appear in the expansion of $(x + y)^4$ really do come from the fourth row of Pascal's triangle. We find ourselves counting ways to arrange 4 x's and y's.

$$(x+y)^4 = \underset{\substack{\uparrow \\ \frac{4!}{4!0!}}}{1x^4} + \underset{\substack{\uparrow \\ \frac{4!}{3!1!}}}{4x^3y} + \underset{\substack{\uparrow \\ \frac{4!}{2!2!}}}{6x^2y^2} + \underset{\substack{\uparrow \\ \frac{4!}{1!3!}}}{4xy^3} + \underset{\substack{\uparrow \\ \frac{4!}{0!4!}}}{1y^4} \dots$$

We can now expand brackets. This result is called the binomial theorem, which states that to expand $(x + y)$ to the n^{th} power, just follow the numbers in the n^{th} row of Pascal's triangle for the coefficients.

People tend to forget in algebra classes that x and y can actually be numbers! For example, let's choose $x = 10$ and $y = 1$ in our formula for $(x + y)^4$. Then, $x + y$ is $10 + 1$, which is 11, so the left side of the formula is 11 to the fourth power. And the right side gives an answer written in powers of 10.

$$\begin{aligned}(x + y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4\\ (10 + 1)^4 &= 1 \times 10^4 + 4 \times 10^3 \times 1 + 6 \times 10^2 \times 1^2 + 4 \times 10 \times 1^3 + 1 \times 1y^4\\ 11^4 &= 1 \times 10{,}000 + 4 \times 1000 + 6 \times 100 + 4 \times 10 + 1\\ &= 14{,}641\end{aligned}$$

We see that 11^4 is 1 times 10,000 plus 4 times 1000 plus 6 times 100 plus 4 times 10 plus 1—that is, this formula is speaking the language of a $1 \leftarrow 10$ machine. 11^4 is 14,641, with the numbers indeed matching the fourth row of Pascal's triangle.

Because every expansion formula from the binomial theorem matches a row of Pascal's triangle, putting in $x = 10$ and $y = 1$ shows that every power of 11 matches a row of the triangle. Mystery explained!

In fact, we can also explain our observation that the entries in the rows of a Pascal's triangle sum to the powers of 2: 1, 2, 4, 8, 16, 32, 64, and so on.

For example, work with $(x + y)^4$ again, and this time put in $x = 1$ and $y = 1$.

$(1 + 1)^4$, which is 2^4, or 16, is 1 times some 1s, 4 times some 1s, 6 times some 1s, 4 times some 1s, and 1 times some 1s. This is $1 + 4 + 6 + 4 + 1$. The sum of the entries of the fourth row of Pascal's triangle is 2^4.

$$\begin{aligned}(x + y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4\\ (1 + 1)^4 &= 1 \times 1^4 + 4 \times 1^3 \times 1 + 6 \times 1^2 \times 1^2 + 4 \times 1 \times 1^3 + 1 \times 1y^4\\ 2^4 &= 1 + 4 + 6 + 4 + 1\end{aligned}$$

Now you can make your own new observations about Pascal's triangle. For example, put in $x = 100$ and $y = 1$ and discover a pattern with the powers of 101. We see the entries of Pascal's triangle encoded as double digits in these powers: 101^0 is 1; 101^1 is 101; 101^2 is 10,201; 101^3 is 1,030,301; and so on.

$101^0 = 1$
$101^1 = 1\ 01$
$101^2 = 1\ 02\ 01$
$101^3 = 1\ 03\ 03\ 01$
$101^4 = 1\ 04\ 06\ 04\ 01$
$101^5 = 1\ 05\ 10\ 10\ 05\ 01$
$101^6 = 1\ 06\ 15\ 20\ 15\ 06\ 01$

FURTHER EXPLORATION

WEB

Tanton, "Permutations and Combinations."
http://gdaymath.com/courses/permutations-and-combinations/

READING

Green, *Pascal's Triangle*.
Tanton, *Thinking Mathematics! Vol. 8*.

PROBLEMS

1. What is the middle entry of the 10th row of Pascal's triangle?

2. In how many different ways can 4 Rs and 10 Ds be arranged so that no 2 Rs are adjacent?

3. What is the coefficient of $x^2y^3z^5$ when we expand $(x + y + z)^{10}$ and collect all like terms?

SOLUTIONS

1. The entry on the 10^{th} row, 5 places in from the left and 5 places in from the right, is $10!/(5!5!) = 252$.

2. This is tricky. One way to think about it is to imagine the 10 Ds in a row with spacers between and either end of them. There are 11 spacers.

_D_D_D_D_D_D_D_D_D_D_

We have 4 Rs to position into those spacers. (This will ensure that no 2 Rs are adjacent.) Because 4 of the spacers will be "used" and 7 will be "not used," there are thus $11!/(4!7!) = 330$ arrangements of the Rs and Ds with no 2 Rs adjacent.

3. The expression $x^2y^3z^5$ will appear in expanding $(x + y + z)^{10}$ from each term in the expansion composed of 2 x's, 3 y's, and 5 z's (such as *xxyyyzzzzz*, *zyyxzzyzzx*, and so on). There are $10!/(2!3!5!) = 2520$ such terms. This, then, is the coefficient of $x^2y^3z^5$.

VISUALIZING RANDOM MOVEMENT, ORDERLY EFFECT

LECTURE 16

In 1905, statistician Karl Pearson used a simple mathematical idea to understand the infestation pattern of mosquitos in a forest: Given a population of mosquitos in a particular location, he assumed that each mosquito will move a fixed distance in some randomly chosen direction at each time step—for example, over each hour of a day. He was then hoping to model the overall effect of the many individual mosquitos and predict the rate of spread of the mosquitoes throughout a forest. Pearson made up the name "random walk" for the motion of an individual guided by random choices. In this lecture, you will learn how to use Pascal's triangle to understand the chaotic behavior of random walks.

RANDOM WALKS

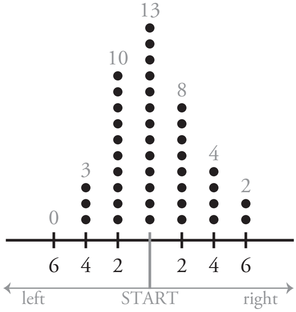
The diffusion of a gas works this way. Individual gas molecules each jitter and jerk about in random directions. So, each individual gas molecule is following a random, unpredictable path. Yet physicists are able to model the overall rate of diffusion of gasses when released into a room: The rate of spread of the gas can be predicted because the aggregate effect of many individual random processes does indeed have predictable overall structure.

To make the mathematics tractable, let's look at random motion on a 1-dimensional line—mosquitoes spreading out along a number line or gasses moving just in 1-dimensional space. We can use the flip of a coin to create the random motion of an individual object. A person can be that object, starting in 1 spot and walking a line according to a flip of a coin: If the coin lands heads, the person takes a step forward. If it lands tails, the person takes a step backward. Then, this is repeated.

If the person were to start over and do the whole walking process again, starting from the same spot, chances are that he or she would end up at a different position along the line—and a different location again if he or she did it a third time. The person's motion is random, and we cannot know where he or she will end up after 6 tosses, for example.

But matters are different if we look at a large number of people each doing their own random movements together at the same time. New Jersey teacher Ralph Pantozzi created a random walk lesson to conduct with school students, in which each student flipped a coin 6 times and recorded his or her results. Then, all of the students performed their individual walks simultaneously. They then lined themselves up against the gym wall so that the counts of people that ended up at each position along the line could be seen. There was the following distribution of people: 13 people ended up back at the starting position, 8 people ended up 2 places to the right of the starting position, 10 landed 2 places to the left of the starting position—and so on.

16.1



The majority of people ended back near or on the starting position. Two people ended up 6 places all the way to the right. With just 6 flips of the coin, they must have received 6 tails in a row. That can—and did—happen. But it is rare. Nobody got 6 heads in a row to end up 6 places to the left.

It is curious that everyone ended up an even number of steps to the left or right of the starting position.

In the distribution pattern of where students ended up, there is some structure—it is a bell-shaped pattern. And Pascal's triangle explains this distribution pattern.

Each student flipped a coin 6 times. How many different sequences of 6 flips are possible? In other words, how many different results might we see among the students when they each flip a coin 6 times?

There are 2 possible outcomes for the first flip (heads or tails), 2 choices for the second flip (heads or tails), 2 for the third flip, and so on. By the multiplication principle for counting, 6 flips correspond to 6 tasks, with 2 options for each task, which makes for $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ possibilities.

So, each student could have seen 1 of 64 possible outcomes. But this doesn't mean that the students could have ended up in 64 different places. For example, someone who tosses HHHTTT will end up back at the starting position—3 steps forward and then 3 steps back. This will also be the case for the person who tosses TTTHHH—3 steps back and then 3 steps forward. In fact, any student who tosses 3 heads and 3 tails in some order ends up back at the starting position. How many arrangements of 3 heads and 3 tails are there that will get a student back to the starting position? There are $6!/(3!3!) = 20$.

$$\frac{\text{3 heads}}{\text{3 tails}} \longrightarrow \frac{6!}{3!3!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1} = 20$$

So, 20 of the 64 possible sequences of tosses have the student returning to the starting position. Therefore, $^{20}/_{64}$, or about 31%, of the students should end back at the beginning. There were 40 students taking part in the experiment, and 31% of 40 is 12.5, so something like 12 or 13 students should end up back at the starting position—and we see that. In fact, 13 students ended back at the beginning position.

About how many students should end up 2 places to the left, for example?

How would a student end two places to the left? Because a forward step negates a backward step, and vice versa, for a student to end up 2 places to the left, he or she must have tossed 2 more heads than tails—4 heads and 2 tails.

How many sequences are there of 4 heads and 2 tails? There are 6!/(4!2!), which works out to be 15.

$$\frac{\text{4 heads}}{\text{2 tails}} \longrightarrow \frac{6!}{4!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1} = 15$$

So, under ideal conditions, we should expect 15⁄64 of the 40 students, or 9.4 of them, to end up 2 places to the left. In fact, there were 10.

We can now work out all the counts of students we expect at each final position by counting all the possible sequences with a given number of heads and tails (**figure 16.2**).

16.2

pattern	6 H	5 H, 1 T	4 H, 2 T	3 H, 3 T	2 H, 4 T	1 H, 5 T	6 T
location	6 L	4 L	2 L	START	2 R	4 R	6 R
count	$\frac{6!}{6!0!} = 1$	$\frac{6!}{5!1!} = 6$	$\frac{6!}{4!2!} = 15$	$\frac{6!}{3!3!} = 20$	$\frac{6!}{2!4!} = 15$	$\frac{6!}{1!5!} = 6$	$\frac{6!}{1!6!} = 1$
percentage	$\frac{1}{64} \approx 1.6\%$	$\frac{6}{64} \approx 9.4\%$	$\frac{15}{64} \approx 23.4\%$	$\frac{20}{64} \approx 31.2\%$	$\frac{15}{64} \approx 23.4\%$	$\frac{6}{64} \approx 9.4\%$	$\frac{1}{64} \approx 1.6\%$
approx. no. from 40 students	0.6	3.8	9.4	12.5	9.4	3.8	0.6

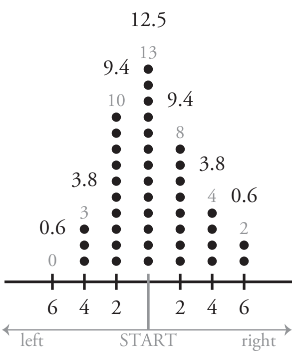
To end up 4 places to the right, toss 1 head and 5 tails. There are 6!/(1!5!) = 6 ways to do this. So, 6⁄64, or 9.4%, of the students should get this. And 9.4% of 40 students is close to 4 students.

The driving force behind the distribution of students is the set of numbers 1, 6, 15, 20, 15, 6, and 1, which are the entries of the sixth row of Pascal's triangle. We first got Pascal's triangle from counting paths in a square grid—that is, counting words of Rs and Ds—and we're counting words of heads and tails. It is the same mathematics.

So, of the 64 possible sequences of tosses, we expect, in ideal circumstances, the count of students landing at each position to be in the proportions 1, 6, 15, 20, 15, 6, and 1 sixty-fourths of the whole group of students.

Ralph Pantozzi's experiment with 40 students closely matches the ideal results we computed. (See **figure 16.3**.)

16.3



If you want to study the diffusion rates of gasses—in our 1-dimensional example, at least—all the answers lie in Pascal's triangle: To see the overall distribution of gas molecules you expect after 6 steps of time, look at the sixth row of Pascal's triangle. To see the overall distribution of gas molecules after 10 units of time, look at the tenth row of Pascal's triangle. After 100 units of time, look at the 100^{th} row. The mathematical diffusion of 1-dimensional gasses is all encoded in Pascal's triangle.

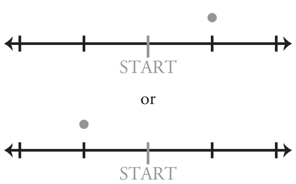
And you can go beyond 1 dimension. For example, if you want to look at the spread of mosquitos in a 2-dimensional forest plane, you can try the analogous work for 2-dimensional motion: Roll a 4-sided object and step east, west, north, or south, according to the roll. The math gets trickier to analyze with this extra freedom of movement, but the ideas are philosophically the same.

If you want to study 3-dimensional diffusion, do the same thing with a 3-dimensional motion: Roll a 6-sided object and accordingly step east, west, north, south, up, or down. The mathematics gets trickier still, but we can at least see an approach for studying aggregate effects of individual random motions.

RETURNING TO THE STARTING POSITION

16.3

16.4



Imagine that an individual gas molecule starts at the initial position on a number line.

The molecule will make a first move, either left or right. Then, it will be 1 off from the starting position. How likely is it that the gas molecule will return to the starting position ever again?

As soon as a molecule steps off of the starting position, it will, with absolute certainty, again return to the starting position at some later time. Then, what happens on the next move? The molecule will step off the starting position again and be 1 unit away. And with absolute certainty, that molecule will return to the starting position yet again at some later time. And the molecule will do this over and over again.

Therefore, if you let a 1-dimensional gas diffuse over an infinite amount of time, every molecule of that gas will return to the starting position infinitely often. Furthermore, every gas molecule actually visits every possible location on the number line infinitely often.

It is possible to count paths in 2-dimensional motion, too, just like we did in our 1-dimensional work. In doing this work, mathematicians have proved that the same behaviors happen in 2-dimensional random walks.

But surprisingly, matters are very different in 3-dimensional random walks. With 6 directions of motion—north, south, east, west, up, and down—there is too much freedom of movement, and the mathematics changes. The probability of an individual molecule ever returning to its starting position in a 3-dimensional random walk is not certain. In fact, mathematicians have computed that there is only about a 34% chance of a molecule in 3 dimensions ever returning to its starting position, even if given an infinite amount of time.

FURTHER EXPLORATION

WEB

Pantozzi, "Take a Field Trip."
http://regionals.nctm.org/wp-content/uploads/AC_Pantozzi-_TakeAFlipTripLessonPlan.2.pdf

Rycroft, "Lecture 1."
http://ocw.mit.edu/courses/mathematics/18-366-random-walks-and-diffusion-fall-2006/lecture-notes/lec01.pdf

READING

Tanton, *Solve This!*

———, *Thinking Mathematics! Vol. 8*.

PROBLEMS

1. The probability of tossing a head with an unbiased coin is 50%. Thus, in tossing the coin 10 times, we might most likely expect to see 5 heads and 5 tails (and if a large number of people each tossed a coin 10 times, this would indeed be the most common outcome). But individual random events are not predictable. What are the chances of actually seeing 5 heads and 5 tails in tossing a coin 10 times?

2. A student tosses a coin 6 times in a row. With each toss of a head, the student steps 1 place to the left, and with each toss of a tail steps 1 place to the right. After 6 tosses, why is the student sure to be an even number of steps from the starting position? (0 is an even number.)

3. Alistair is standing in 1 cell of a grid of squares. He spins a spinner that will land on 1 of the words "north," "south," "east," "west"—each equally likely to appear. Alistair then steps 1 square over in the direction given by the spinner. He then repeats this process over and over again, spinning the spinner and making a step, to thus perform a 2-dimensional random walk on the square grid.

a What are the chances that Alistair will return to the starting position after just 2 steps?

b What are the chances that Alistair will return to the starting position after 4 steps?

4. Rebecca will toss a coin until the total count of heads she has seen is greater than the total count of tails. (So, for example, she will stop after 1 toss if she first tosses heads or stop after 3 tosses if she tosses tails, heads, and then heads.) Is she certain to eventually stop tossing? (Assume that she is willing and able to toss an infinite number of times if necessary.)

SOLUTIONS

1. There are $10!/(5!5!) = 252$ ways to create a list of 5 Hs and 5 Ts in a row, so there are this many ways to see 5 heads and 5 tails tossed. There are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024$ possible outcomes in total for tossing a coin 10 times in a row. Thus, the probability of seeing 5 heads and 5 tails is $252/1024 \approx$ 24.6%.

2. Because a left step and right step cancel their effects, in any list of 6 coin tosses, we can cross out a head and a tail and be left with a shorter list of steps for the student to actually follow. (The steps from HHTHTH, for example, lands the student at the same place as HHTH, which is equivalent to just HH.) Thus, in effect, a student follows a path of steps away from the starting position that is either 6 steps, 4 steps, 2 steps, or 0 steps long.

3. **a** There are $4 \times 4 = 16$ possible results that Alistair could see from spinning the spinner 2 times. Of those, only the outcomes NS, SN, EW, and WE return him to the starting position (4 possibilities). The chances of him returning to the starting position after 2 steps is $4/16 = 25\%$.

b There are $4 \times 4 \times 4 \times 4 = 256$ possible results that Alistair could see from spinning the spinner 4 times. Of those, only those outcomes with 2 Ns and 2 Ss (there are $4!/(2!2!) = 6$ of these); 2 Es and 2 Ws (there are $4!/(2!2!) = 6$ of these); or 1 E, 1 W, 1 N, and 1 S (there are $4! = 24$ of these) have him return to the starting position (36 possibilities in total). The chances of him returning to the starting position after 4 steps is $36/256 \approx 14.1\%$.

4. This scenario matches a random walk on a number line; a head corresponds to stepping left and a tail corresponds to stepping right. Because every random walk is certain to visit every position on the number line, eventually the number –1 will be reached. Here, the coin-tossing game stops, because this means that 1 more head has appeared than tails. The game is thus certain to stop. (We just cannot predict when!)

VISUALIZING ORDERLY MOVEMENT, RANDOM EFFECT

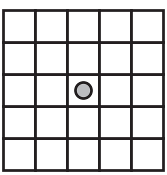
LECTURE 17

The previous lecture examined how random, chaotic behavior can exhibit, in aggregate effect, overall structure and some predictability. In this lecture, you will examine the reverse phenomenon: how a completely known and predictable motion leads to a seemingly chaotic and unpredictable global effect. Two visually striking examples—one with simple 2-dimensional grid motion and one with simple 1-dimensional paper folding—will help you understand the balance between predictable behavior and complex chaotic or infinitely delicate behavior at a global level. And that global level might surprise you and have regular structure to it, too.

LANGTON'S ANT

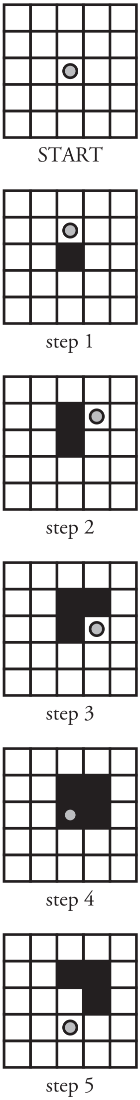
In 1986, computer scientist Christopher Langton wondered to what extent automated phenomena could represent, or at least mimic, organic life. In his explorations, he invented a very simple computer simulation that is today known as Langton's ant. The construct is very simple.

17.1



An ant lives on a 2-dimensional array of squares, a big array extending infinitely far in all directions, and all the ant does is walk from cell to neighboring cell. Initially, all the cells are colored white, but the ant will change the colors of cells as it moves along. It starts in some location, facing upward.

17.2



Here are the rules of the ant's motion: First, the ant changes the color of the cell it is in—from white to black, or later, as the game is played, from black to white—and then it takes a step forward to a new cell. If the cell it lands in is white, the ant will turn right 90°. If it is black, it will turn left 90°. Then, it does all this over again, and again, and again.

So, the ant starts facing upward. Its first move is to change the color of the cell it is in, step forward, and turn right 90° because it just entered a white cell. So, after its first step, it leaves a black cell behind and is facing right.

For the next move, the ant changes the color of the cell it is in, takes another step forward, and turns right because the new cell is again white. So, after 2 steps, it has left behind 2 black cells and is now facing downward.

This is it for the ant's motion: Change the cell color, step forward, and turn either left or right, according to the color of the new cell. The ant's third step takes it to another white cell, and it turns right again. The ant's fourth step takes it to a black cell, and it turns left for the first time. The fifth step is to another white cell—and so on.

The ant's motion is completely fixed, determined, and predictable. There is nothing random about this setup and structure. Where the ant shall be and what it shall be doing on the millionth step is predetermined.

If we let the ant just do its thing over hundreds, even thousands, of steps with the help of computer simulation, the ant eventually falls into a pattern of predictable motion.

Turner Bohlen wrote a computer animation for Langton's ant (available at www.turnerbohlen.com/langtonsant/), and you can run and play with his program. You can set up a 70 × 70 grid of squares, all white. Remember that this is meant to be an infinite array of cells, so when the ant gets to the edge of the display, it will behave differently than what Christopher Langton described. You can make different choices about how the ant handles being on an edge. However, for a 70 × 70 display, the ant doesn't touch the edge for a long time. Running the ant on this finite board shows exactly what the ant would do on an infinite board—until it reaches the edge.

If we start with an all-white grid with the ant starting in a cell facing upward, the first few thousand steps of the ant's motion looks random and unpredictable—pure chaos. Even though the rules for the motion are completely rigid, the long-term results are very difficult to understand. In fact, mathematicians can say very little about the structure of this long-term motion—it is just too chaotic!

Figure 17.3 shows the chaotic pattern of the ant's motion after 8000, 9000, and 10,000 steps.

17.3

≈8000 steps ≈9000 steps ≈10,000 steps

17.4

But something shocking occurs between the 10,000th and 11,000th steps: The ant suddenly falls into a pattern of predictable motion. It builds a regular "highway" heading off to infinity in the southeast direction. No one understands why!

This behavior truly is a mystery to mathematicians. No one has ever found an example of where it does not, and no one can currently explain or justify why it always should!

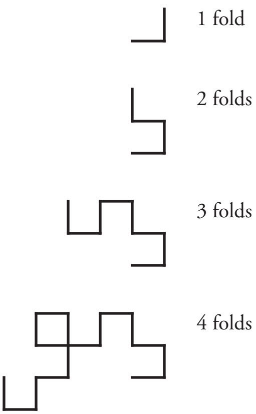
Langton's ant seems like a double paradox: It is a system based on very structured, rigid, predictable rules that, despite our intuition, seem to lead to apparently chaotic, random behavior. And just as our mind adjusts to that chaotic reality, Langton's ant shocks us by behaving in a structured, nonrandom way.

Langton's ant typifies a new type of visualization power in mathematics. It is only with the advent of computers—capable of running tens of thousands of steps of computation and showing pictures along the way—that mathematicians would even notice this curious behavior of the ant or even think there was something to study in its automated motion.

DRAGON CURVES

This second example is another example of predictable step-by-step behavior leading to surprisingly complex structure. This one we can do by hand to see both chaos and structure side by side. It's about paper folding.

17.5



Fold a strip of paper in half once and open it to a 90° bend, and we get an L shape. Keep folding and we get the shapes shown in **figure 17.5**.

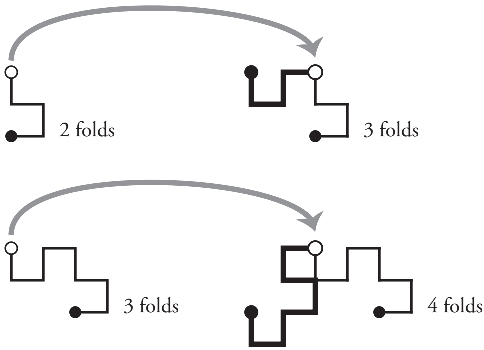
In the geometry of these pictures, each curve is just a copy of the previous curve plus another copy of that previous curve rotated 90°. And these 2 curves are joined at a left-turning bend.

We can draw these pictures very easily on a computer: Draw the beginning picture, make a copy of that picture, rotate it 90°, and then add the rotated copy to the original. Then, repeat this process. (See **figure 17.6**.)

That sounds simple enough. But there might be a problem.

Look at the 4-fold picture. The folded paper just touches itself at a corner. This makes us wonder if it is actually possible to keep going with this construction—might we ever get into a pickle with the rotated copy of the picture crossing over the original copy? This would be a serious problem for physical paper. It would mean that we are trying to force the paper to pass through itself, which is impossible.

17.6

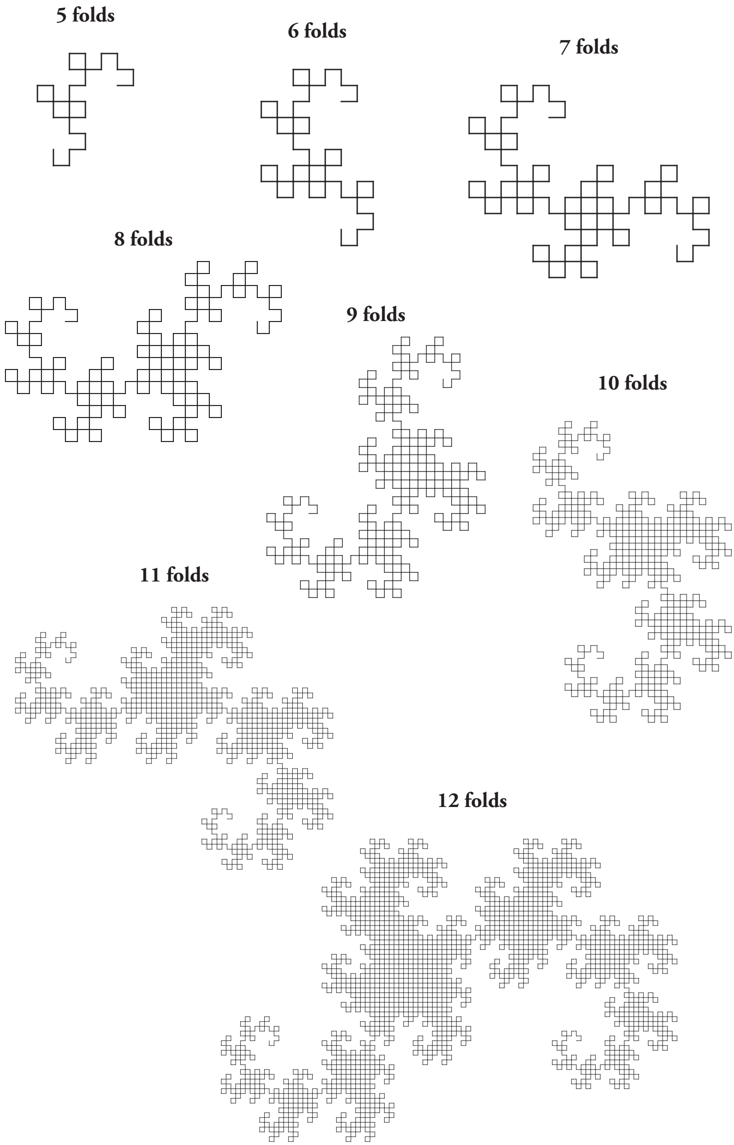


The amazing thing is that this construction never gets into a pickle. The pictures we make will never intersect through each other, and as for the physical work, the paper will always give itself enough space to properly open up at a 90° bend for every fold. It might touch at corners, but it will never be stymied by trying to pass through itself.

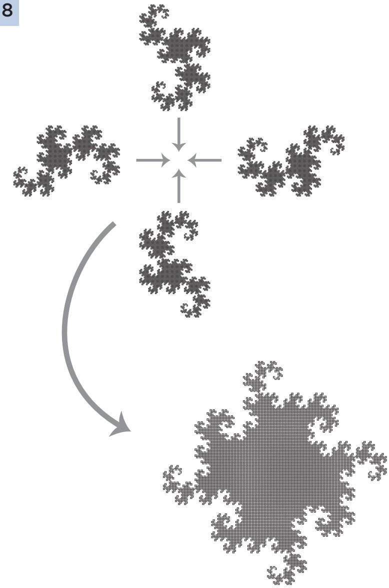
Knowing that we can keep going, let's keep going! **Figure 17.7** on the following page shows the pictures of 5, 6, 7, 8, 9, 10, 11, and 12 folds.

A beautiful fractal-like curve appears. The 12-fold pattern looks like it is made of sections that are each a smaller copy of the whole picture, and that's what mathematicians mean by a fractal—an image that contains scaled copies of itself. If we kept going with this process, we'd get more and more jagged pictures with more and more scaled copies appearing within themselves. The fractal curve being formed in this process is called the dragon curve by mathematicians.

17.7


5 folds
6 folds
7 folds
8 folds
9 folds
10 folds
11 folds
12 folds

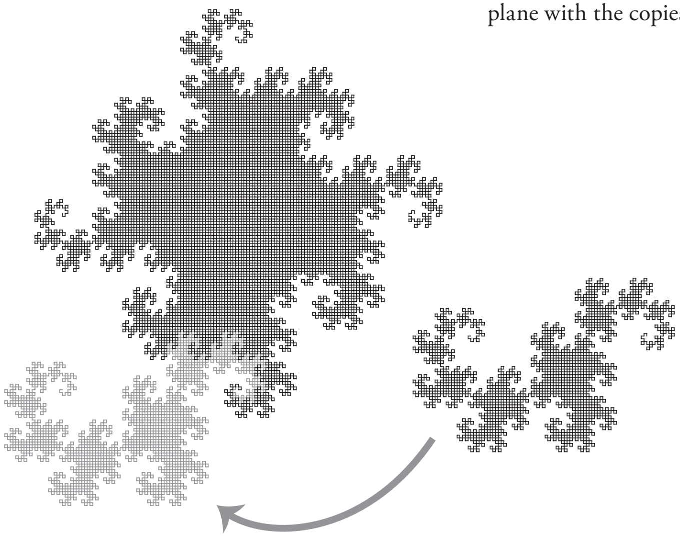
17.8



This curve is geometrically quite astounding. For example, take 4 copies of the curve, oriented in the 4 different directions: north, south, east, and west. Despite their jaggedness, they all interlock beautifully without overlap. All the fractal edges line up in magical perfection.

And if we want, we can fit in more copies of the dragon curve into the design, again perfectly, without overlap and without gaps where the edges line up. (See **figure 17.9**.)

In fact, we can keep fitting in more and more copies of the dragon curve and tile the whole plane with the copies of the curve.



17.9

FURTHER EXPLORATION

READING

Bailey, Kim, and Stritcharts, "Inside the Lévy Dragon."
Gallivan, *How to Fold Paper in Half Twelve Times*.
Langton, "Studying Artificial Life with Cellular Automata."
Propp, "Further Ant-ics."
Tanton, *Mathematics Galore!* (Explore why the dragon curve tiles the plane.)
———, *Solve This!* (Explore why Langton's ant cannot be bounded within a finite region of the plane.)

PROBLEMS

1. It is about 93 million miles to the Sun. If we fold a very long strip of paper, just 0.001 inches thick, in half multiple times, how many folds would give us a wad of paper tall enough to reach the Sun?

2. An ant moves from cell to cell, either vertically or horizontally, in a square grid. It alternates taking vertical and horizontal steps. If an ant entered 1 particular cell from a vertical direction, could it later reenter that cell from a horizontal direction?

SOLUTIONS

1. The distance 93 million miles equals $93{,}000{,}000 \times 5280 = 491{,}040{,}000{,}000$ feet, which equals 5,892,480,000,000 inches. If each layer of paper is 0.001 inches thick, this corresponds to 5,892,480,000,000,000 layers of paper.

Folding a piece of paper in half 53 times produces $2^{53} = 9{,}007{,}199{,}254{,}270{,}992$ layers. Just 53 folds is thus enough to produce a stack that reaches the Sun.

2. For any motion in a square grid, a path that returns to the starting position must have an equal number of left and right horizontal steps and an equal number of up and down vertical steps. Thus, there must be an even number of steps in any loop.

   If the ant is required to alternate in vertical and horizontal steps, then after completing an even number of steps, the ant is back to moving either vertically or horizontally, as it first started. So, if the ant first entered a cell in a vertical direction, it is always set to reenter that cell from a vertical direction thereafter.

VISUALIZING THE FIBONACCI NUMBERS

LECTURE 18

In this lecture, you will explore a very famous sequence of numbers in mathematics—the Fibonacci numbers—and get a sense of why these numbers have fascinated mathematicians and math enthusiasts for centuries. The mathematical structures and patterns in this sequence are astounding and deep, and people are still discovering new results about the Fibonacci numbers. You will discover a single picture that explains everything, as well as a new result of the Fibonacci numbers.

THE FIBONACCI NUMBERS

Up to the 12^{th} century, scholars, merchants, and everyday people in Europe were using the Roman numeral system for counting: I for stood for 1, V for 5, X for 10, L for 50, and so on. This system is fine for recording numbers; it's really just a tally system. The number 67, for example, in the Roman system is LXVII:

I = 1
V = 5
X = 10
L = 50
C = 100
D = 500
M = 1000

But mathematical thinking and doing was really held back by this Roman system. Try adding together 67 and 94 using only the Roman symbols. It is very difficult!

$$\begin{array}{r} \text{LXVII} \\ +\ \text{CLXXXXIIII} \\ \hline ? \end{array}$$

In the latter part of the 12th century, Italian mathematician Leonardo Fibonacci traveled extensively through northern Africa and the Middle East. He noticed that scholars and merchants there were using a different notational system for numbers; they were using place value and just 10 symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. He observed that with this notational system, arithmetic and mathematics were natural, easy, and fruitful.

So, Fibonacci wrote a book, *Liber abaci*, the book of counting, which was released in 1202. The goal of the book was to explain the Hindu-Arabic numeral system to the European populous and provide practice examples on using the system for doing arithmetic. Fibonacci's book was extraordinarily influential, and it led the Western world to let go of Roman numerals and move to the number system we all use today.

There was one practice problem that Fibonacci included in his book that caught the fancy of many scholars at the time—and after that time, and to this day! It is a problem that leads to the famous sequence of numbers that bears his name.

Here's the practice problem Fibonacci devised:

How many rabbits would be produced in the n^{th} month if, starting from a single pair, any pair of rabbits of 1 month produces 1 pair of rabbits each month after the next?

We start with a single pair of rabbits the first month.

The question says that after 1 month, any pair of rabbits produces 1 pair of rabbits each month after the next. This means that the pair matures for a month and then produces a new pair of rabbits every month thereafter. So, in month 2, nothing happens; the pair is still maturing.

In month 3, they produce a new pair of rabbits (the gray pair).

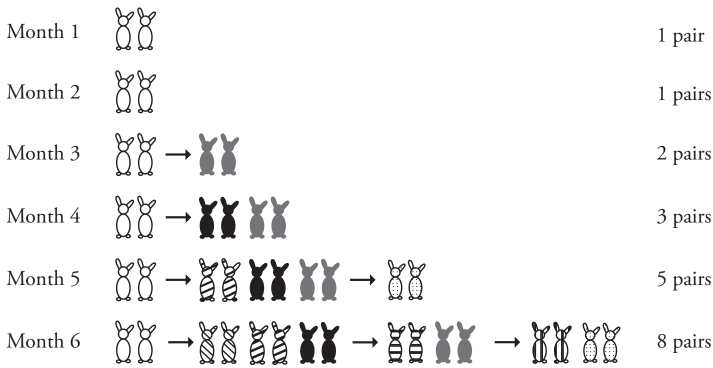
In month 4, the white pair produces yet another pair; they breed every month from now on. Suppose that they produce a black pair. The gray pair, on the other hand, is still maturing.

In month 5, the white rabbits produce yet another pair. The gray rabbits are mature and produce a pair, and the black rabbits are still maturing.

In month 6, the white rabbit, gray rabbit, and now black rabbit pairs that were around in month 4 are mature enough to each produce a new pair. And we still have all rabbits that were present in month 5.

In fact, the count of rabbits in any 1 month is all the rabbits you had the previous month and a new pair for each pair present 2 months ago—that is, the count of rabbits in any 1 month is the sum of counts from the 2 previous months.

18.1



We see that 1 + 1 is 2, 1 + 2 is 3, 2 + 3 is 5, 3 + 5 is 8. Next month there will be 5 + 8, or 13 rabbits. Then, there will be 8 + 13, or 21 rabbits, and so on.

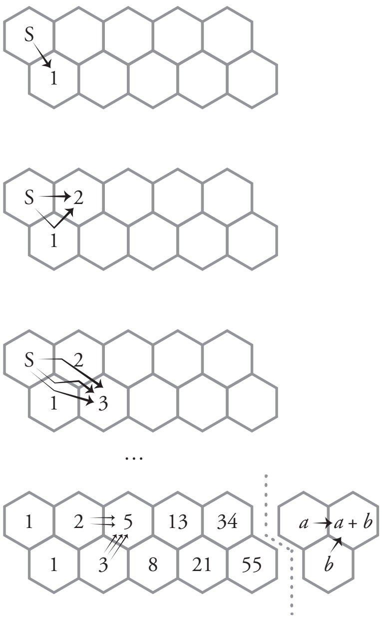
We get the sequence of Fibonacci numbers. Each number, after the initial pair of ones, is the sum of the 2 numbers just before it.

> The Fibonacci Numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, …

These particular numbers captured the fascination of some many scholars and math enthusiasts because they keep coming up over and over again in the most unexpected places. The Fibonacci numbers have been spotted in tens, if not hundreds, of different contexts.

THE HONEYCOMB PICTURE

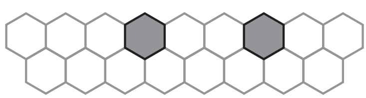
18.2



There is a single picture of a honeycomb that explains many of the mysterious appearances of the Fibonacci numbers. The count of paths that always move rightward in a honeycomb is always a Fibonacci number. If we start at the leftmost cell, then the number of allowable paths to the second cell is 1. The number of allowable paths to the third cell is 2. The number of allowable paths for the fourth cell is 3, and then 5, and then 8, and so on.

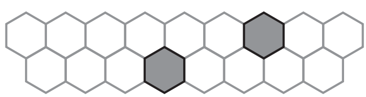
And to match the Fibonacci numbers precisely, we'll say that there is 1 way to move to the start cell: just stand there!

18.3



More generally, we saw that the number of paths between any 2 cells in the honeycomb is a Fibonacci number. For example, there are 7 cells in the zigzag connecting the 2 positions that are shaded gray in **figure 18.3**. Thus, the number of paths between these 2 cells is the seventh Fibonacci number: 13.

18.4



As another example, there are 6 cells in the zigzag connecting the 2 gray positions in **figure 18.4**. So, there are 8 paths between them.

THE MULTIPLICATIVE STRUCTURE OF THE NUMBERS

The Fibonacci numbers have a mathematical feature that helps explain why mathematicians are fascinated by them.

The Fibonacci numbers are defined in an additive way: We start with 2 ones, and each number thereafter is the sum of the 2 numbers just before it: 1 + 1 is 2, 1 + 2 is 3, 2 + 3 is 5, 3 + 5 is 8, and so on. **Figure 18.5** is the list of the first 15 Fibonacci numbers.

18.5

Place:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Fibonacci Number:	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Shockingly, it turns out that these numbers have a multiplicative structure, as well.

For example, the number 12 is a multiple of 6. The 12th Fibonacci number is 144. The sixth Fibonacci number is 8, and 144 is a multiple of 8.

18.6

Place:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Fibonacci Number:	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

$$144 = 8 \times 18$$

The 15th Fibonacci number is 610. 15 is a multiple of both 3 and 5. The 15th Fibonacci number is a multiple of both the third and fifth ones.

18.7

Place:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Fibonacci Number:	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

$$610 = 2 \times 305$$
$$610 = 5 \times 122$$

The 16th Fibonacci number is 377 + 610 = 987. 16 is a multiple of 2, 4, and 8. And 987, the 16th Fibonacci number, is a multiple of each of the second, fourth, and eighth ones.

18.8

Place:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Fibonacci Number:	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

$$987 = 1 \times 987$$
$$987 = 3 \times 229$$
$$987 = 21 \times 47$$

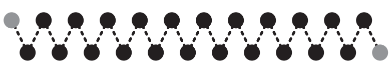
It turns out that if n is a multiple of a, then the n^{th} Fibonacci number is sure to be a multiple of the a^{th} Fibonacci number.

Why would a simple additive sequence have such a consistent and deep multiplicative structure? This result can be proven true, using pages of algebra. But *why* is the result true?

If we think about honeycombs, we can explain this multiplicative property with a picture. We can even give complete formulas for all that is going on!

Let's do a specific example that reveals the general approach. We know that 24 is a multiple of 8. We don't know what the 24th Fibonacci number is, but we can prove that it must be a multiple of the eighth Fibonacci number, which is 21. The 24th Fibonacci number must be a multiple of the eighth Fibonacci number: 21.

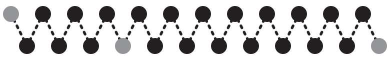
18.9



F_{24} = the 24th Fibonacci number
= number of paths from gray dot to gray dot

The 24th Fibonacci number is the count of paths along cells in a honeycomb. Let's write F_{24} for the 24th Fibonacci number—whatever it is.

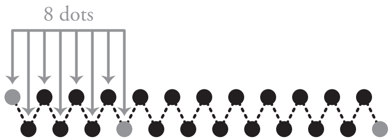
18.10



Because we are interested in the number 8 and its multiples, we are going to ask questions about the eighth dot in this picture, as well as the 16th dot and the 24th dot.

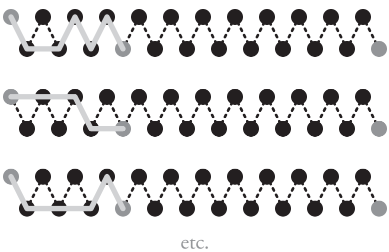
Here's the first question about the eighth dot: How many paths from start to finish hit the eighth dot? In other words, how many paths visit each of the 3 gray dots?

18.11



To create such a path, we have 2 tasks: Fill in a path for the left section of the diagram, and fill in a path for the right section of the diagram. The left section, from gray dot to gray dot, is a segment of 8 dots.

18.12

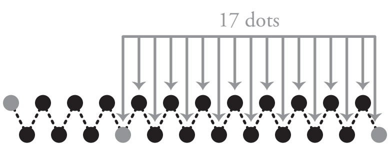


F_8 possible paths in all

So, the number of paths we could draw in the left section is the eighth Fibonacci number, F_8, which is 21.

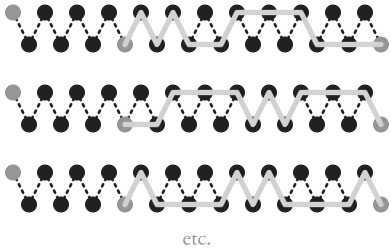
The right section of the honeycomb, from gray dot to gray dot, is 17 dots long. (We are using that middle dot again, hence the count of 17 and not 16 as you first might expect.)

18.13



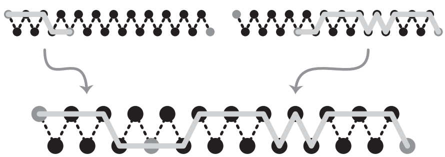
There are F_{17} paths we could choose to fill in the right section.

18.14

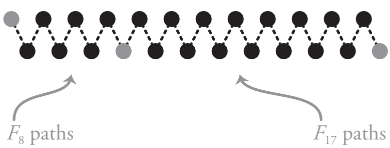


F_7 possible paths in all

18.15

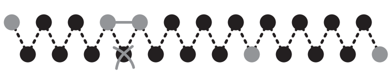


18.16



$F_8 \times F_{17}$ paths in all

18.17



Choose a path for the left piece. Choose a path for the right piece. We see in **figure 18.15** that they combine to give a full path that does indeed pass through the eighth dot.

Because there are F_8 possible choices for a left path and F_{17} choices for a right path, then by the multiplication principle of counting, there are thus F_8 times F_{17} ways to make a path that goes through the eighth dot. This answer is a multiple of F_8, the eighth Fibonacci number.

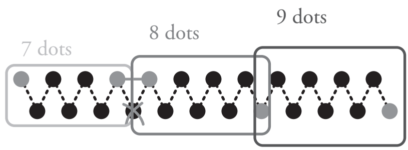
We are trying to show that the 24^{th} Fibonacci number is a multiple of the eighth Fibonacci number. To do this, we're focusing on the dots that are multiples of 8.

We just showed that there are F_8 times F_{17} paths that go through the eighth dot.

How many paths miss the eighth dot but go through the 16^{th} dot instead?

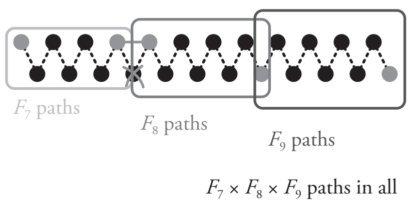
A path that misses the eighth dot but hits the 16^{th} must break into 3 sections, as shown in **figure 18.17**.

18.18



This first section has 7 dots. The middle section has 8 dots. The last section has 9 dots.

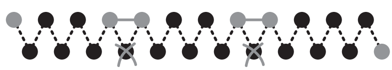
18.19



The number of possible paths we can draw for the first section is F_7, the seventh Fibonacci number. The number of paths we can draw in the middle section is F_8, the eighth Fibonacci number. The number of paths we can draw in the last section is F_9, the ninth Fibonacci number. This means that there are $F_7 \times F_8 \times F_9$ ways to complete this picture—that is, to fill in the 3 sections and make 1 complete path along the full honeycomb. This answer is a multiple of F_8, too.

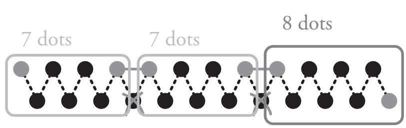
The number of paths that miss the eighth dot but hit the 16^{th} dot is also a multiple of the eighth Fibonacci number.

18.20



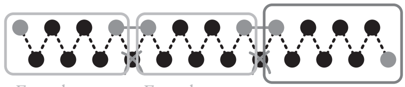
How many paths miss both the eighth and the 16^{th} dots? Such a path must have the structure shown in **figure 18.20**.

18.21



It, too, breaks into 3 sections: a left section that is 7 dots long, a middle section that is 7 dots long, and a final section that is 8 dots long.

18.22



$F_7 \times F_7 \times F_8$ paths in all

We see that are $F_7 \times F_7 \times F_8$ ways to fill in the picture to make a full path. This answer is also a number that is a multiple of F_8, the eighth Fibonacci number.

Every path from beginning to end must fall into 1 of these 3 categories: It either hits the eighth dot, or it misses it and hits the 16th dot instead, or it misses both completely and just ends at the 24th dot, like all these paths do.

So, the total number of paths in the entire diagram must be the sum of the 3 counts we got for each of these 3 cases. Because each case gave a count that was a multiple of F_8, the eighth Fibonacci number, the sum is itself a multiple of F_8.

We just proved that the count of paths along the entire honeycomb is a multiple of the eighth Fibonacci number. But we know the total number of paths along 24 dots in a honeycomb: It's the 24th Fibonacci number. So, actually, we have just proved that the 24th Fibonacci number is a multiple of the eighth one, just as we hoped to do.

This is really quite surprising. Recall that the Fibonacci numbers are defined by an additive process—each is the sum of the previous 2. However, they seem to have this astounding property of respecting multiplication: 24 is a multiple of 8, and the 24th Fibonacci number is a multiple of the eighth Fibonacci number.

If you kept track of the formulas we computed, we actually proved this formula:

$$F_{24} = F_8 \times F_{17} \times F_7 \times F_8 \times F_9 \times F_7 \times F_7 \times F_8$$

From this, we can obtain a formula for what you actually get when you divide the 24th Fibonacci number by the eighth one. It looks like this:

$$\frac{F_{24}}{F_8} = F_{17} \times F_7 \times F_9 \times F_7 \times F_7$$

And this result seems to be new to the world.

You can use the idea of the honeycomb proof to establish the general formula for the quotient of Fibonacci numbers:

If n is a multiple of a, then

$$\frac{F_n}{F_a} = F_{n+1-a} + F_{a-1}F_{n+1-2a} + (F_{a-1})^2 F_{n+1-3a} + \ldots + (F_{a-1})^{n/a} F_1$$

You can even get a formula for the cases when remainders are involved.

FURTHER EXPLORATION

WEB

Tanton, "Fibonacci Surprises." http://www.jamestanton.com/wp-content/uploads/2009/04/Fibonacci-Surprises_Tulsa-March-20161.pptx.

READING

Ball, *Strange Curves, Counting Rabbits, and Other Mathematical Explorations*.

PROBLEMS

1. The sequence of Fibonacci numbers begins 1, 1, 2, 3, 5, 8, 13, 21, 34…. Here, each number, after the second, is the sum of the 2 numbers just before it.

If we follow this pattern, is it possible to extend the Fibonacci numbers to the left? What is the appropriate value to place to the left of the 2 ones? What is the appropriate value to place to the left of that?

2. People studying number theory are interested in counting the number of ways to break a given number into sums. For example, the number 2 can be written just as 2 or as 1 + 1. The number 3 can be written just as 3 or as 1 + 2 or 2 + 1 or 1 + 1 + 1. And sometimes a mathematical structure might offer more than 1 option for particular items. For example, **figure 18.23** shows the partitions of some numbers using 2 different types of 1.

These counts appear to be Fibonacci numbers!

Use paths in a honeycomb to explain why the count of ways to break a given number into a sum of terms, with 2 types of 1 available, is sure to be a Fibonacci number.

18.23

1	2 ways
1	
2	5 ways
1 + 1	
1 + 1	
1 + 1	
1 + 1	
3	13 ways
2 + 1	
2 + 1	
1 + 2	
1 + 2	
1 + 1 + 1	
1 + 1 + 1	
1 + 1 + 1	
1 + 1 + 1	
1 + 1 + 1	
1 + 1 + 1	
1 + 1 + 1	
1 + 1 + 1	

18.24

2	1 way
3	1 way
4 2 + 2	2 ways
5 2 + 3 3 + 2	3 ways
6 2 + 4 4 + 2 3 + 3 2 + 2 + 2	5 ways

3. How many ways are there to write a given number greater than 1 as a sum of terms never using the digit 1? Is the answer always a Fibonacci number?

18.25

1	1 way
1 + 1	1 way
3 1 + 1 + 1	2 ways
1 + 3 3 + 1 1 + 1+ 1	3 ways
5 1 + 1 + 3 1 + 3 + 1 3 + 1 + 1 1 + 1 + 1 + 1 + 1	5 ways

4. How many ways are there to write a given number greater than 1 as a sum of odd terms? Is the answer always a Fibonacci number?

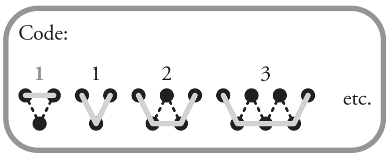
SOLUTIONS

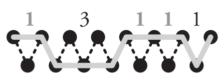
1. It is possible to extend the Fibonacci sequence infinitely far to the left and preserve the property that each term is the sum of the 2 terms before it.

…, −21, 13 −8, 5 −3, 2 −1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, …

2. Each path between 2 cells on the top row of a honeycomb matches a partition of a number with 2 types of 1. Because there are a Fibonacci number of paths, there are a Fibonacci number of partitions.

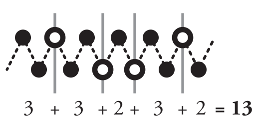
18.26





3. The count is indeed always a Fibonacci number.

18.27

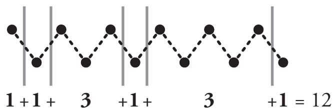


For any given path in the honeycomb, circle the cells it misses. This breaks the segments of the zigzag line between the start and end cells into sections that are at least 2 segments long. If we imagine an extra segment placed at the start and end of the diagram as well, then the beginning and end sections of the zigzag are sure to be at least 2 segments long, as well. We get a partition of the total number of segments present in the diagram into a sum with each term at least 2.

And, conversely, given a partition into terms 2 or larger, we can reconstruct the path whence it came. Thus, the count of 1-less partitions matches the count of paths—a Fibonacci number.

4. The count is indeed always a Fibonacci number.

18.28



For any path in the honeycomb, imagine the diagonal steps as breaking the nodes of the zigzag path into sections. Each section must contain an odd number of nodes, and we get a partition of the total count of nodes present in the diagram into a sum of odd terms.

And, conversely, given a partition of this number into odd terms, we can reconstruct the path whence it came. Thus, the count of odd-only partitions matches the count of paths—a Fibonacci number.

THE VISUALS OF GRAPHS

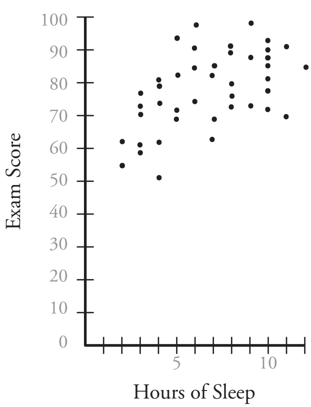
LECTURE 19

In this lecture, you will learn about graphing. Specifically, you will be introduced to graphs from equations, which are scatterplots of data values that make the equation true, and graphs from functions, which are scatterplots of input/out data from a function. You will also be exposed to a third source of data to plot: sequences. Drawing graphs as you might know them from math class can be a powerful alternative visual tool in mathematics.

GRAPHS

Graphs are visual ways to represent and compare data. In fact, every graph—every 2-dimensional graph—is really a plot of 2 data categories compared side by side.

19.1

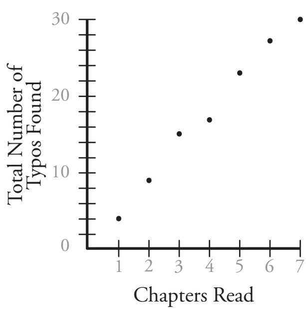


For example, imagine that a professor wonders if there is any correlation between students' sleeping habits and their performance on an exam. The professor suspects that the more sleep a student gets before a test, the better he or she does. The professor gives an exam, including this question: How many hours of sleep did you get last night? The professor then grades all the exams and compares the exam scores with the answers to that question. **Figure 19.1** shows a plot of the class results.

There are 2 axes for the 2 data variables the professor is interested in: the horizontal axis for the number of hours of sleep each student got, and the vertical axis for the exam scores out of 100. Each point in the plot represents the 2 values of a single student: exam score and number of hours of sleep.

When the professor looks at the plot, it might appear that students who got more sleep did, overall, a little better than those who got less sleep. But a correlation is not clear. In addition, other factors might affect exam scores—not just hours of sleep the night before. Perhaps the number of hours of study students put into preparing for the exam is a prime factor, for example. Although the graph offers no clear information, it is useful nonetheless in helping the professor research this question about exam performance.

19.2



Data in a plot might indeed suggest a strong correlation between 2 factors. For example, in writing a new book, the writer asks someone to proofread the manuscript. **Figure 19.2** is a graph showing the total number of typos the proofreader found compared with the number of chapters worked through.

The proofreader found 4 typos after reading the first chapter, a total of 9 typos after reading 2 chapters, a total of 15 typos after reading the third chapter, and so on.

Actually, the shape of this graph makes sense: As you read more chapters, the total count of typos found only increases. But what's interesting about this graph is that it looks like the data points are following, approximately, a straight-line path. It seems that with each new chapter read, the count of errors found increases, more or less, by the same amount. The conclusion is that the writer makes about the same number of typos in every chapter.

Statisticians call plots like these—points comparing 2 types of data categories—scatterplots. And the pure visuals of scatterplots allow you to see trends or dismiss conjectured trends.

GRAPHING EQUATIONS

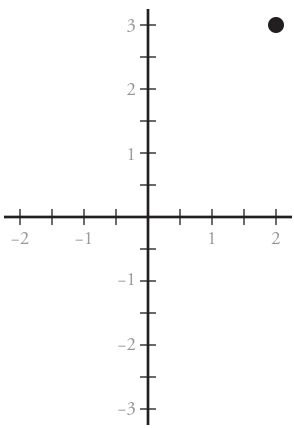
In math classes, the idea of graphing seems to have a different flavor to it. Plotting data and scatterplots seems to belong to statistics, and graphing in a math class is all about graphing equations and curves and functions—which has nothing to do with plotting data.

But that's actually not quite right! All the graphing done in math class is actually data plotting, too. People just tend not to say that—or think that.

For example, here's an equation:

$$a^3 + 1 = b^2$$

19.3

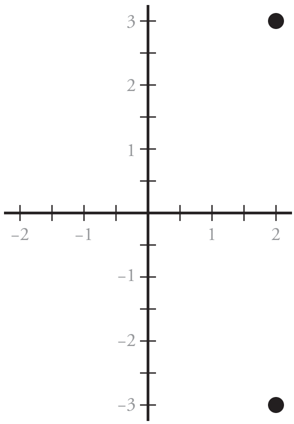


Is there any natural data to collect from an equation? What seems to be the natural data to associate with the equation $a^3 + 1 = b^2$.

The natural data to collect is the set of all values of a and b that make the equation true. For example, $a = 2$ and $b = 2$ works: $2^3 + 1$ is $(2 \times 2 \times 2) + 1$—that is, $8 + 1$, which is 9. And this does equal 3 squared: $2^3 + 1 = 3^2$.

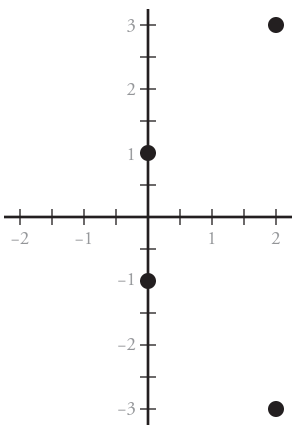
We have a data point to plot: $a = 2$, $b = 3$.

19.4



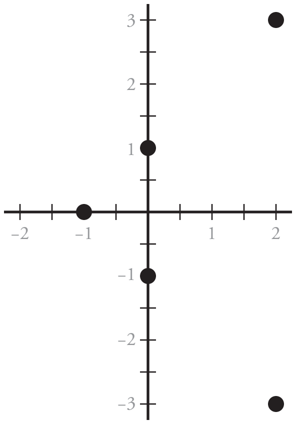
Actually, we also have $2^3 + 1 = -3^2$. So, we have the point (2, −3), too.

19.5



There are also 2 data points with $a = 0$: $0^3 + 1 = 1^2$ and $0^3 + 1 = -1^2$. We have the points ($a = 0$, $b = 1$) and ($a = 0$, $b = -1$) to add to the plot.

19.6



There is also a data point with $b = 0$ and $a = -1$: $(-1)^3 + 1 = 0^2$.

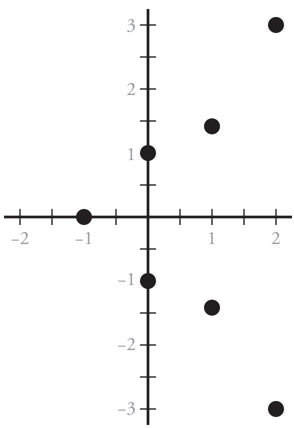
Now it is getting difficult to find more whole-number solutions. In fact, mathematicians have proven that there are no more whole numbers that fit this particular equation. But we don't need to stick with integers. For example, put in $a = 1$, and we get $1^3 + 1 = b^2$.

This shows that $b = \sqrt{2}$ or b is its negative version, $-\sqrt{2}$. We get 2 more data points.

$$1^3 + 1 = (\sqrt{2})^2$$
$$1^3 + 1 = (-\sqrt{2})^2$$

$\sqrt{2}$ is about 1.4, so these 2 data points plot about here: (1, 1.4) and (1, −1.4).

19.7

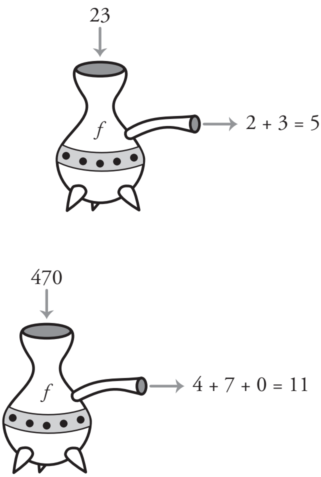


In fact, we can keep plugging in different values for a, work out what b has to be for each of those values, and plot more and more data points for this equation. As you do this, the shape of its graph gradually emerges.

And because we can use a whole continuum of a values, it is appropriate to say that there is a whole continuum of data points—that is, the scatterplot for the equation $a^3 + 1 = b^2$ will really be the continuum of a curve in the plane. A graph of an equation is a scatterplot of data obtained from that equation.

GRAPHING FUNCTIONS

19.8



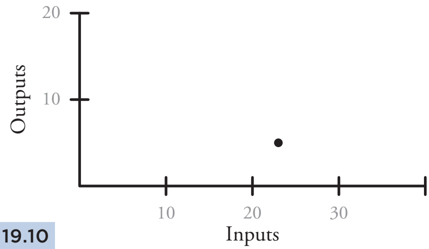
In math classes, you also learn about functions. A function is simply a rule that associates an output with each input from some given set of inputs. In other words, a function is some kind of machine: Put in an input, and the machine churns out an associated output.

For example, consider the function that associates with each counting number the sum of its digits. To the number 23, assign the number 2 + 3, which is 5. To the number 470, assign the number 4 + 7 + 0 = 11—and so on. That's a function: We have a rule that assigns an output to each input that is the sum of the input's digits. (See **figure 19.8**.)

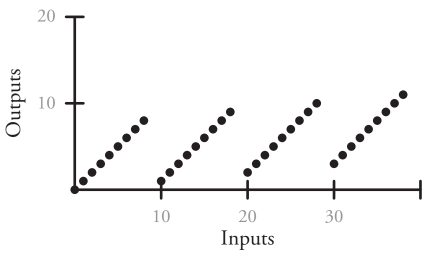
What's the natural data to collect from this function? The input/output pairs seem to be natural. The pair 23 and 5, for example, is a natural data point. We can plot that.

And we can plot a whole host of data points and get the scatterplot for this function. It looks like **figure 19.10**.

19.9



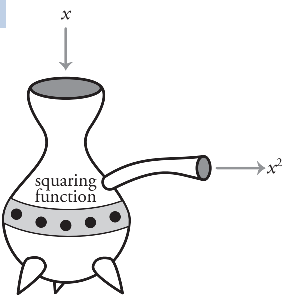
19.10



The letters x and y have a favored role in math classes. The letter x is often used to denote the input value of a given function, and y is the associated output value. People write an equation in the variables x and y and call that equation a function. But calling an equation a function is a bit strange.

Look at the squaring function, for example, the rule that assigns to each number that number squared. For example, associate to the input 5 the output 25. Associate to the input –3 the output 9. Associate to the input 1.5 the output 2.25—and so on. Just square the input.

19.11



In other words, associate to an input value x the output value x^2.

The natural data to plot for this function would be the pairs 5 and 25, –3 and 9, 1.5 and 2.25, and so on—the input/output pairs.

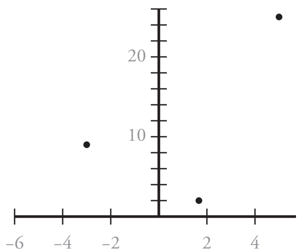
Now consider the equation $y = x^2$.

What is the natural data to associate with this equation? It's the x and y values that make the equation true. For example, $x = 5$ and $y = 25$ works, and $x = -3$ and $y = 9$ works, and $x = 1.5$ and $y = 2.25$ works. The equation $y = x^2$ gives exactly the same data as the squaring function and therefore will give exactly the same scatterplot.

Philosophically, we have 2 different things here—the scatterplot of a function and the scatterplot of an equation—but they turn out to be identical, and people usually don't bother holding the distinction in their minds. People tend to just think of x as input and y as output, so the equation $y = x^2$ is seen as a rule for a function—each output is just the input squared—and each rule for a function in math class is seen as an equation.

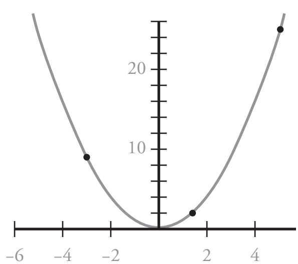
However you want to think about it, we have a scatterplot. We currently have 3 data points: (5, 25), (–3, 9), and (1.5, 2.25) for the squaring function. Let's plot them. (See **figure 19.12**.)

19.12



If we plot more and more data points, we get a symmetric U-shaped curve for the formula $y = x^2$. And because the inputs can come from a whole continuum of values, the outputs cover a whole continuum of values, too. We thus get a connected continuum of data points—that is, the plot is a curve.

19.13



GRAPHING SEQUENCES

A third source of data to plot in the typical math class is sequences. Here's the sequence of Fibonacci numbers:

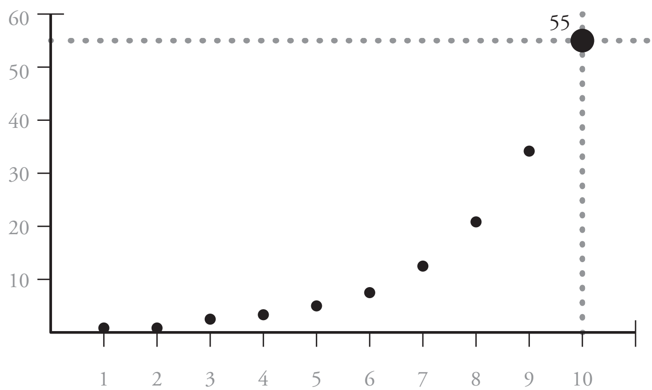
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, …

What data could we associate with a sequence? The first number of the Fibonacci sequence is 1, the second number of the sequence is 1, the third number is 2, the fourth number is 3, the 12th number is 144, and so on.

Let's associate with each counting number n the Fibonacci number in that n^{th} position. The sixth Fibonacci number is 13, so we have the data point (6, 13). This seems natural.

Here's the scatterplot of the first 10 Fibonacci numbers. We can see from this graph that the 10th Fibonacci number is 55, for example.

19.14



We now have a sense of the shape of the Fibonacci numbers. They grow in size, and the rate of growth seems to increase.

FURTHER EXPLORATION

WEB

Tanton, "Curriculum Essay: October 2015."
http://www.jamestanton.com/wp-content/uploads/2012/03/Curriculum-Essay_October-2015_On-Graphing.pdf

READING

Cleveland, *Visualizing Data*.

PROBLEMS

1. Consider the function that assigns to each counting number its last digit. Sketch a graph of this function.

2. Over your life span, consider your height at different moments of time. How might a graph of your height-versus-time data appear?

3. Here is a classic puzzle:

   At 7 a.m., a monk starts at the base of a mountain and climbs to its peak, arriving there at 7 p.m. His route up the mountain was far from uniform: He paused to look at sights along the way, climbed into gullies and back out again, and so on.

   After spending the night at the peak, he headed back down the mountain the next day at 7 a.m., taking a different route. He arrived back at the base of the mountain at 7 p.m.

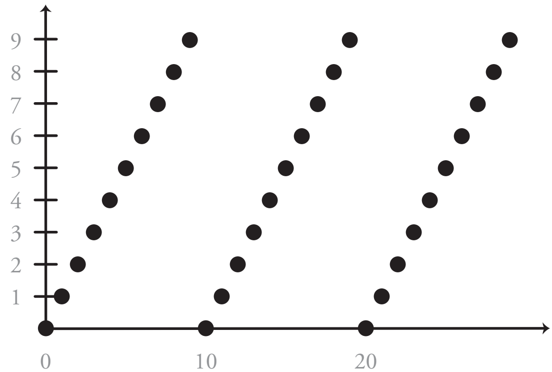
   Prove that there was a time strictly between 7 a.m. and 7 p.m. for which the monk was at exactly the same height at that moment on both days.

4. Sketch a graph of all the points (x, y) in the plane that make the equation $x^2 + y^2 = 1$ true. What is the graph of all points (x, y, z) in 3-dimensional space that make $x^2 + y^2 + z^2 = 1$ true?

SOLUTIONS

19.15

1.

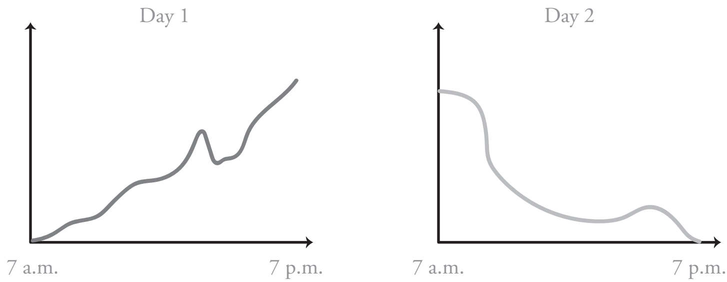


2. The answer varies for each individual. Let's say that a certain 50-year-old man stopped growing around age 20, and his height has decreased slightly over the decades.

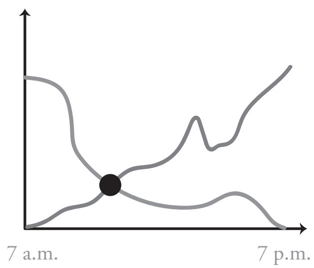
19.16

3. Draw graphs of the monk's height on the first day and on the second day.

19.17



19.18



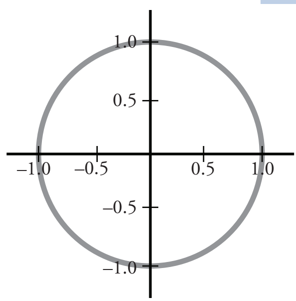
Now place the graphs of top of each other to see that there must have been a time of day at which the monk's heights matched.

Regardless of the shapes of the graphs, there must be a point at which they intersect.

4. The graph of $x^2 + y^2 = 1$ is a circle of radius 1. Points such as $(1, 0)$, $(3/5, -4/5)$, and $(-21/29, -20/29)$ lie on this circle.

The graph of $x^2 + y^2 + z^2 = 1$ in 3-dimensional space is a sphere of radius 1.

19.19



SYMMETRY: REVITALIZING QUADRATICS GRAPHING

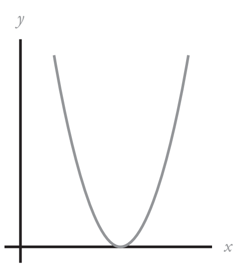
LECTURE 20

Mathematicians love symmetry. If there is symmetry in a mathematical situation, then there is hope that we can make good progress understanding that situation. The goal of this lecture is to completely revitalize the work of quadratics and turn the typical high school experience around. The lecture aims to prove that symmetry is your friend and to show you the absolute power of it even when applied to a difficult algebra course. This lecture will show you how to use visual symmetry to make the graphing of quadratics clear and straightforward.

SYMMETRIC U-SHAPED CURVES

The graph of the equation $y = x^2$ is a symmetrical U-shaped curve.

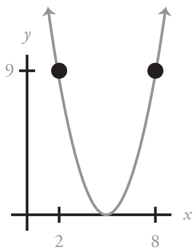
20.1



This curve, and its natural variations—called quadratic equations—are studied in great depth in a typical algebra class.

Suppose that you have a symmetric U-shaped curve, one like that of $y = x^2$, sitting somewhere else on the x-axis. And suppose that you know that this curve passes through the points (2, 9) and (8, 9). Without knowing the formula for the curve, can you tell where the curve touches the x-axis?

20.2

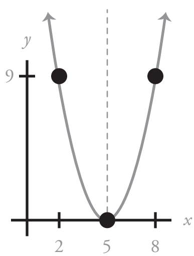


The picture has 2 symmetrical points. In fact, it has 2 symmetrical points sitting on a symmetrical curve.

Common sense tells us that the line of symmetry for the picture must be right between the 2 and the 8 on the x-axis. That occurs at position 5. And now we can see that the curve touches down on the x-axis at position 5. (See **figure 20.3**.)

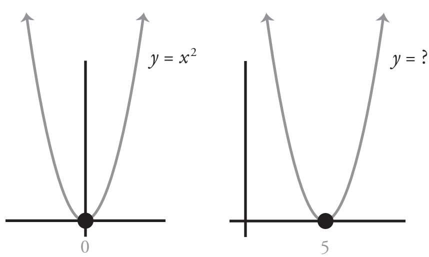
The equation $y = x^2$ is an example of a quadratic equation, and high school algebra classes study both the algebra and the graphing of these equations.

20.3



This example shows how thinking about symmetry can make sense of complex ideas. You don't have to memorize the quadratic formula, for example; you can see it and derive it from playing with the symmetry of a square. And you don't have to have formulas in your head to help you sketch a quadratic graph; all you have to do is look for symmetrical points and then let symmetry guide you the rest of the way.

20.4

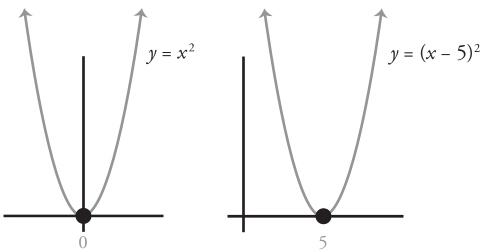


Let's find an equation for the curve that just touches down at 5 on the x-axis.

The only equation we currently have is $y = x^2$ for the U-shaped curve that touches down at $x = 0$. Can we turn this into an equation that touches down at $x = 5$?

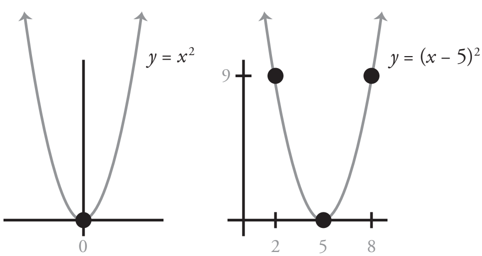
Looking at these 2 pictures side by side, we notice that they are essentially identical. We just have the number 5 in the right picture behaving like 0 in the left picture. Can we rewrite the formula $y = x^2$ so that the number 5 is behaving like 0? Let's try $y = (x - 5)^2$.

20.5



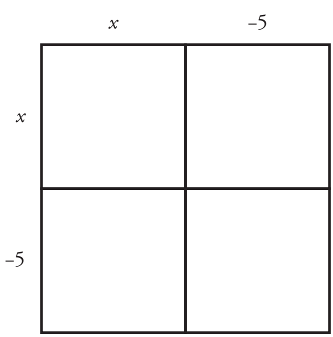
When we put in $x = 5$, we get $y = (5 - 5)^2$, or $y = 0^2$. The number 5 is indeed behaving like 0 here. The graph of $y = (x - 5)^2$ is the symmetrical U-shaped graph that touches down at 5 on the x-axis.

20.6



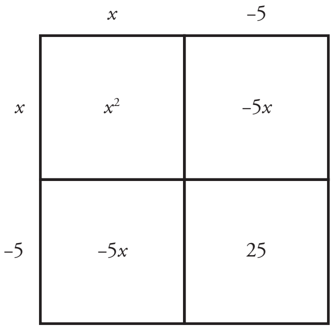
We can check that this formula also passes through the 2 original points. Put in $x = 8$, and we get $y = (8 - 5)^2$, or $y = 3^2$, which is 9. And put in $x = 2$, and we get $y = (2 - 5)^2$, or $y = (-3)^2$, which is also 9.

20.7



But let's go a little bit further. The formula we have is $y = (x - 5)^2$. The algebra here is referring to the geometry of a square—an $(x - 5)$-by-$(x - 5)$ square. If we draw a picture of this square, we see that it naturally breaks into 4 pieces. (Allowing negative quantities in our geometric pictures still represents truth in arithmetic and algebra.)

20.8



$(x - 5)^2 = x^2 - 10x + 25$

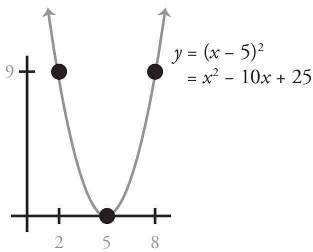
What are the areas of each of the pieces? We have an x-by-x piece of area, which is x^2; a –5-by-x piece of area, which is $-5x$; another one of those; and a piece of area –5 by –5, which is 25.

So, we see that $(x - 5)^2$ is algebraically the same as $x^2 - 5x - 5x + 25$—that is, $(x - 5)^2 = x^2 - 10x + 25$. Thus, we can say that the equation of our graph touching down at 5 on the x-axis is also given by $y = x^2 - 10x + 25$, which is algebraically the same. This is less enlightening as $(x - 5)^2$, but it is nonetheless equivalent.

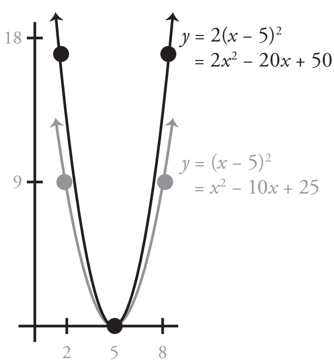
This illustrates the general point about studying quadratics in school. Usually, an equation is given to you in an unenlightening form, and you usually memorize formulas for handling these messy equations. But if you can see your way through to a more enlightened version of the equation, then everything falls into place, with nothing to be memorized!

Matters can become quite messy. For example, we might change the steepness of the curve. For instance, $y = 2(x - 5)^2$ gives a curve whose outputs are twice as big as what they were before.

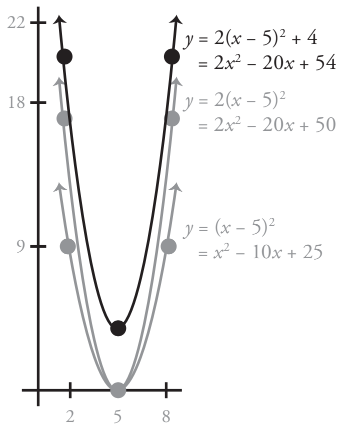
20.9



20.10



20.11



Or we could add 4 to all the output values and get a graph that is 4 units higher. This would have an equation of $y = 2(x - 5)^2 + 4$. (See **figure 20.11**.)

No matter how scary things look, we can cut through all the scariness and use symmetry to see our way through it. In the end, every quadratic equation is of the form $y = ax^2 + bx + c$ for some numbers a, b, and c.

Such a formula comes from playing with the basic quadratic equation $y = x^2$ and transforming it—for example, making 5 behave like 0 to get $y = (x - 5)^2$, doubling all the outputs to get $y = 2(x - 5)^2$, and shifting all the outputs up by 4 to get $y = 2(x - 5)^2 + 4$. And because $y = x^2$ is a symmetrical U-shaped curve to begin with, and all the shifts and transformations we do never change its symmetry, we can argue that the graph of any quadratic equation, no matter how bizarrely it is presented to us, must give a symmetrical U-shaped graph.

EXAMPLE

20.12

	x	-4
x	x^2	$-5x$
-14	$-14x$	56

$(x - 4)(x - 14) = x^2 - 18x + 56$

Sketch the graph of $y = (x - 4)(x - 14)$.

This formula is not of the form we expect. It doesn't look like $ax^2 + bx + c$. But if we expand this expression, it really is an expression of this form, just in disguise. An $(x - 4)$-by-$(x - 14)$ rectangle divides into 4 pieces—x^2, $-4x$, $-14x$, and 56—and we see that $(x - 4) \times (x - 14)$ equals $x^2 - 18x + 56$. We do have a quadratic.

Knowing that, forget the algebra! All we need to know is that its graph is going to be a symmetrical U-shaped curve. If we can find 2 symmetrical points on a symmetrical graph, then everything will just fall into place.

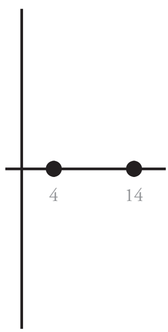
Look at the equation $y = (x - 4)(x - 14)$. Are there any obvious symmetrical points to think about?

20.13

$y = (x - 4)(x - 14)$

$x = 4 \longrightarrow y = 0$

$x = 14 \longrightarrow y = 0$



The numbers 4 and 14 are staring right at us.

If you put in $x = 4$, you get $y = 0 \times (-10)$, which is 0.

If you put in $x = 14$, you get 10×0, which is again 0.

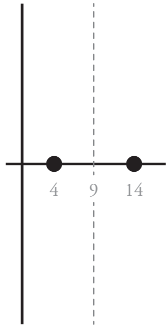
We've found 2 symmetrical points on our symmetrical graph!

20.14

$y = (x - 4)(x - 14)$

$x = 4 \longrightarrow y = 0$

$x = 14 \longrightarrow y = 0$

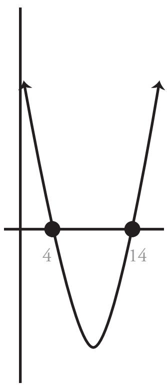


The line of symmetry must be right between 4 and 14 on the x-axis, which is at position 9.

So, we must have a U-shaped graph that looks something like **figure 20.15**.

But there is a problem: How low does the graph go? Is it very low? Is it shallow? How can we tell?

20.15

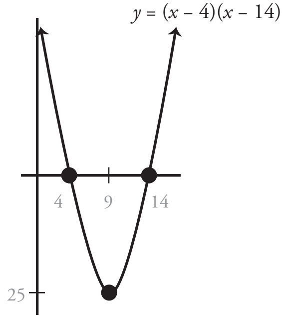


What do we have to go on? What if we put in $x = 9$ into our formula? After all, the lowest point on the curve happens at $x = 9$. When x is 9, y is $(9 - 4) \times (9 - 14)$, or $5 \times (-5)$, which is -25. We now have a good sketch of the curve. The lowest output of the function is -25. (See **figure 20.16**.)

If you want to add more details, you can. For example, you can compute where it crosses the y-axis by putting in $x = 0$. But these kinds of things are just details. We have a great sketch of the quadratic just from locating 2 symmetrical points and using our wits thereafter.

20.16

$$x = 9 \longrightarrow y = (9 - 4)(9 - 14) = (5)(-5) = -25$$



FURTHER EXPLORATION

WEB

Tanton, “Cool Math Essay: October 2014.”
(Explore a connection between quadratic graphs and ordinary multiplication.)
http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_OCTOBER-2014_Parabolic-Multiplication1.pdf

———, “Quadratics.”
http://gdaymath.com/courses/quadratics/.

PROBLEMS

1. **a** Sketch a graph of $y = (x - 14)(x - 20) - 7$.

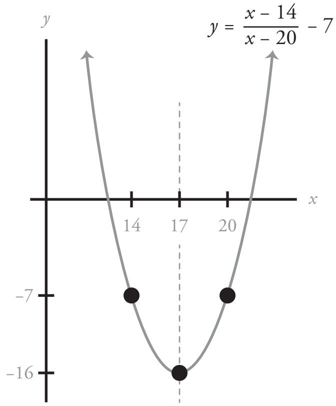
   b Sketch a graph of $y = 3x^2 + 6x + 503$.

2. Find the equation of the quadratic whose graph crosses the x-axis at $x = -8$ and $x = 12$ and has a lowest y-value of -5.

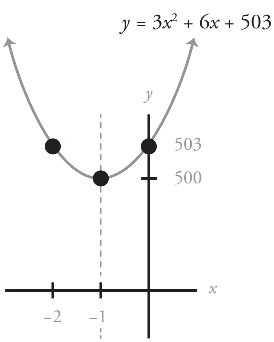
3. Students often have to memorize that the line of symmetry of the quadratic equation $y = ax^2 + bx + c$ occurs at position $x = -b/(2a)$. Derive this formula.

SOLUTIONS

20.17



20.18



1. **a** We see that $x = 14$ and $x = 20$ are interesting x-values. They both give $y = -7$. Thus, we have 2 symmetrical points on the graph. The line of symmetry must be at $x = 17$, and the y-value of the curve at this point is $y = (3)(-3) - 7 = -16$. This is enough information to sketch the curve. (See **figure 20.17**.)

b Writing $y = 3x^2 + 6x + 503$ as $y = 3x(x + 2) + 503$, we see that $x = 0$ and $x = -2$ are interesting x-values. They both give $y = 503$. Thus, we have 2 symmetrical points on the graph. The line of symmetry must be at $x = -1$, and the y-value of the curve at this point is $y = 3(-1)(1) + 503 = 500$. This is enough information to sketch the curve. (See **figure 20.18**.)

2. We are told that $x = -8$ and $x = 12$ both give 0. This suggests a formula of the form $y = a(x + 8)(x - 12)$, where a is some steepness factor. The line of symmetry occurs at $x = 2$, and the function should have a value of -5 there. We get $-5 = a(2 + 8)(2 - 12)$, giving $a = 1/20$. The quadratic we seek is $y = (1/20)(x + 8)(x - 12)$.

3. Write $y = x(ax + b) + c$. This shows that both $x = 0$ and $x = -b/a$ give $y = c$. The line of symmetry must be halfway between these 2 x-values. This is $x = (1/2)(-b/a) = -b/(2a)$.

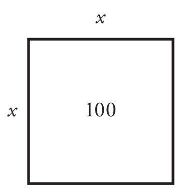
SYMMETRY: REVITALIZING QUADRATICS ALGEBRA

LECTURE 21

The previous lecture focused on the graphing of quadratics but sidestepped the algebra of quadratics. The lecture played around with equations for the purposes of graphing but didn't play around with the equations for the sake of algebra—for finding numerical solutions to problems. You might remember this as a big topic from your school days: learning the quadratic formula for solving equations and completing the square to rewrite expressions. In this lecture, you will learn how to do all of this algebra visually, using the power of symmetry.

EXAMPLE 1

21.1



$x^2 = 100 \longrightarrow x = 10$ or -10

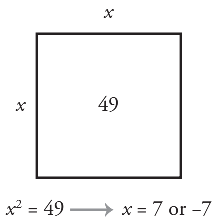
Solve $x^2 = 100$.

This is a geometry problem. What is the side of a square that has an area of 100? A square of side length 10 has an area of 100.

But as an arithmetic problem, there is actually a second number whose square is 100—namely, –10: $(-10)^2 = 100$. So, pushing beyond geometry, we have 2 solutions to the quadratic equation $x^2 = 100$. In the world of arithmetic, $x = 10$ or $x = -10$.

EXAMPLE 2

21.2



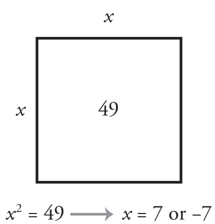
Solve $x^2 = 49$.

We have a square whose area is 49. It's side length must be 7. Going a step beyond geometry, there is a second arithmetical solution: -7.

EXAMPLE 3

Solve $(x - 4)^2 = 144$.

21.3



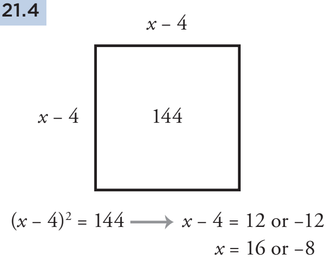
Something squared is 144, and that something must be 12 or -12, so $x - 4$ is either 12 or -12.

Adding 4 throughout, we see that x must be $12 + 4$, which equals 16, or $-12 + 4$, which equals -8. The equation therefore has solutions of 16 or -8.

EXAMPLE 4

Solve $(x + 3)^2 = 17$.

21.4



Something squared is 17. Because 17 is an awkward number, something squared is 17 means that something is $\sqrt{17}$ or its negative version, $-\sqrt{17}$.

The something is $x + 3$, so $x + 3$ is $\sqrt{17}$ or $-\sqrt{17}$. Subtract 3 throughout to see that x is either $\sqrt{17} - 3$ or $-\sqrt{17} - 3$.

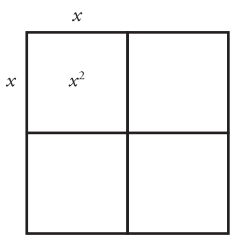
The numbers are quite awkward this time, but the ideas are the same. If we are told the area of the square, then we can solve the problem.

EXAMPLE 5

Solve $x^2 + 6x + 9 = 64$.

21.5

$x^2 + 6x + 9 = 64$



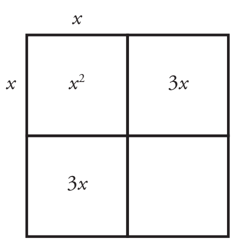
This looks scary, but it is probably an easy problem in disguise. We're hoping it is of the form "something squared is 64"—that is, we're hoping that $x^2 + 6x + 9$ comes from 4 pieces of a square.

The x^2 term must be 1 piece, coming from x times x.

We know that 2 pieces combine to make $6x$. Because we want to be working with a square—we should keep things symmetrical—we need 2 equal pieces: $3x$ and $3x$.

21.6

$x^2 + 6x + 9 = 64$

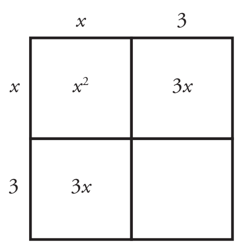


We can now start filling in some details of the square. Something times x gives $3x$; we must have a side piece of 3.

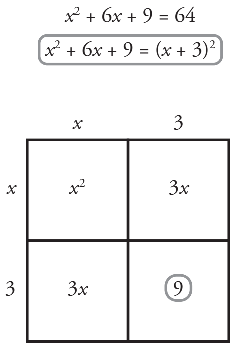
Now that we have the sides of length 3, this means that the area of the final piece must be 3 times 3, which is 9. And 9 is precisely the number we are looking for!

21.7

$x^2 + 6x + 9 = 64$



21.8



So, we now see that $x^2 + 6x + 9$ is really an $(x + 3)$-by-$(x + 3)$ square in disguise. It's really $(x + 3)^2$.

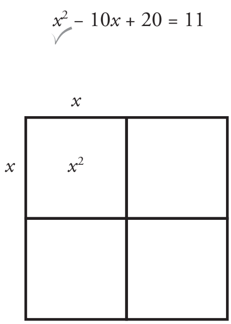
So, the problem we've been asked to solve, $x^2 + 6x + 9 = 64$, is really just the problem $(x + 3)^2 = 64$ in disguise.

$$x^2 + 6x + 9 = 64$$
$$(x + 3)^2 = 64$$
$$x + 3 = 8 \text{ or } -8$$
$$x = 5 \text{ or } -11$$

And we know how to solve that! Something squared is 64. That something better be 8 or −8—that is, $x + 3$ is either 8 or −8. To get x, subtract 3 to see that x is either 8 − 3, which equals 5, or −8 − 3, which equals −11.

EXAMPLE 6

21.9

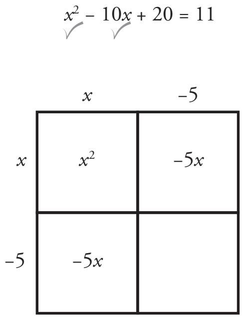


Solve $x^2 - 10x + 20 = 11$.

This must also be a straightforward problem in disguise. It must come from a square divided into 4 pieces. There is an x^2 piece at the very least coming from x times x.

The $-10x$ must come from 2 pieces. To keep things symmetrical, the pieces must be $-5x$ and $-5x$.

21.10

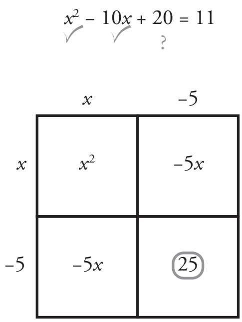


Something times x gives $-5x$. We must be dealing with the number -5 in the remaining side portions of the square.

This means that the final region of the square is -5×-5, which is 25.

And now we have a problem. We got the number 25, but the question wants the number 20. There is a mismatch.

21.11

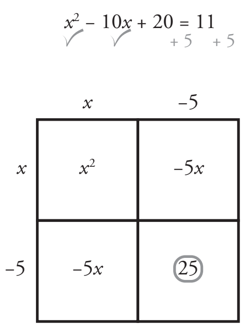


Wouldn't it be nice if the problem had a 25 instead of a 20? If there is something you want in life, make it happen—and deal with the consequences.

Let's make it a 25 by adding 5 to the left side of the equation. And as a consequence of that action, we must also add a 5 to the right side to keep things balanced.

So, the equation we need to solve is now $x^2 - 10x + 25 = 16$.

21.12



What is $x^2 - 10x + 25$? It is all the pieces of our square—our $(x - 5)$-by-$(x - 5)$ square. So, we see that this is really the equation $(x - 5)^2 = 16$ in disguise.

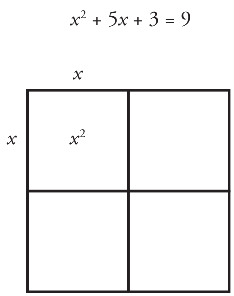
$$x^2 - 10x + 20 = 11$$
$$x^2 - 10x + 25 = 16$$
$$\boxed{(x - 5)^2 = 16}$$

And we can solve that. Something squared is 16. So, the something is either 4 or −4—that is, $x - 5$ is either 4 or −4. Adding 5 throughout, we get that x equals $4 + 5$, which is 9, or $-4 + 5$, which is 1. So, x is 9 or 1.

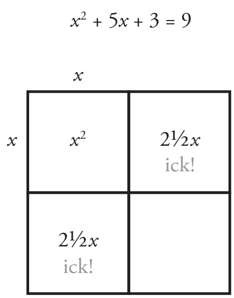
$$x^2 - 10x + 25 = 16$$
$$(x - 5)^2 = 16$$
$$x - 5 = 4 \text{ or } -4$$
$$x = 9 \text{ or } 1$$

EXAMPLE 7

21.13



21.14



Solve $x^2 + 5x + 3 = 9$.

Let's start by drawing the square. We know right away that it has an x^2 piece coming from x times x.

The $5x$ is awkward. To keep the square a square, we need to split this into 2 equal pieces: $2.5x$ and $2.5x$.

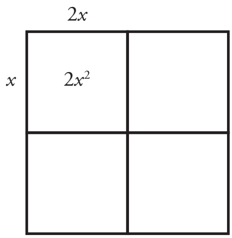
Is there a way to avoid fractions?

The problem is the $5x$: 5 is an odd number. Is there a way to change the equation to avoid an odd number in the middle? One approach would be to multiply everything by 2 and make the equation $2x^2 + 10x + 6 = 18$. That's still the same equation, and now we have $10x$ in the middle—which is much better!

$$x^2 + 5x + 3 = 9$$
$$2x^2 + 10x + 6 = 18$$

21.15

$$x^2 + 5x + 3 = 9$$
$$\times\, 2$$
$$2x^2 + 10x + 6 = 18$$



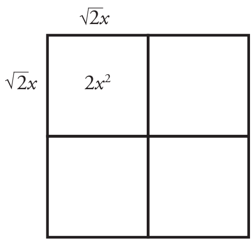
So, let's do the square method on this. The first piece is $2x^2$. This comes from x times $2x$—which ruins the symmetry of the square.

To keep it a square, we need to work with identical numbers. It must be $\sqrt{2}x$ and $\sqrt{2}x$.

Is there a way we can avoid both fractions and square roots?

21.16

$$x^2 + 5x + 3 = 9$$
$$\times\, 2$$
$$2x^2 + 10x + 6 = 18$$



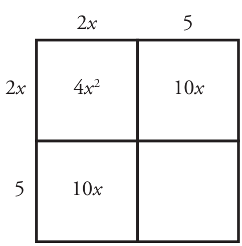
We have $x^2 + 5x + 3 = 9$. We want the middle term to be an even multiple of x to avoid fractions, and we want any numbers that appear with x^2 to keep it a perfect square.

What about multiplying through by 4? Let's make the equation $4x^2 + 20x + 12 = 36$.

Is this a good fit for the square?

21.17

$$x^2 + 5x + 3 = 9$$
$$\times\, 4$$
$$4x^2 + 20x + 12 = 36$$



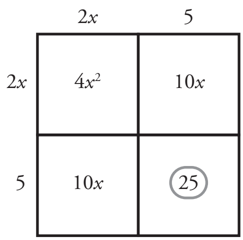
$4x^2$ is good; it comes from $2x$ times $2x$.

The $20x$ is good; it splits as $10x$ and $10x$.

Something times $2x$ equals $10x$. We must be dealing with the number 5 on the sides.

21.18

$$x^2 + 5x + 3 = 9$$
$$\times 4$$
$$4x^2 + 20x + 12 = 36$$
$$+13 \quad +13$$



This means that the magic number we want for the final piece must be 25. But we don't have it—we have a 12.

To make the 12 a 25, let's add 13 to both sides.

So, we now have the equation $4x^2 + 20x + 25 = 49$. And we see from the square that this is just $(2x + 5)^2 = 49$. And we can solve that!

$$4x^2 + 20x + 25 = 49$$
$$(2x + 5)^2 = 49$$

Something squared is 49. So, the something is 7 or −7.

$$2x + 5 = 7 \text{ or } -7$$
$$2x = 2 \text{ or } -12$$
$$x = 1 \text{ or } -6$$

In other words, $2x + 5$ is 7 or −7. Subtract 5 throughout: $2x = 7 - 5$, which is 2, or $2x = -7 - 5$, which is −12. Dividing by 2 now gives $x = 1$ or $x = -6$.

EXAMPLE 8

Solve $3x^2 - 7x - 2 = 8$.

Right off the bat, we are worried about the $3x^2$ at the front. That is not a perfect square; it's $\sqrt{3}x$ times $\sqrt{3}x$, and we want to avoid square roots. What if we multiplied through by another 3? The equation becomes $9x^2 - 21x - 6 = 24$.

$$3x^2 - 7x - 2 = 8$$
$$9x^2 - 21x - 6 = 24$$

Now we have $9x^2$ out front, which is fine—that's $3x$ times $3x$. But we have a problem with the middle term: $-21x$ is odd.

Let's multiply through by 4 to resolve that issue.

$$9x^2 - 21x - 6 = 24$$
$$36x^2 - 84x - 24 = 96$$

21.19

$$3x - 7x - 2 = 8$$
$$\times\ 12$$
$$36x^2 - 84x - 24 = 96$$

	$6x$	-7
$6x$	$36x^2$	$-42x$
-7	$-42x$	

This seems absurd, because all of the numbers are now much bigger, but we're successfully avoiding square roots and fractions. We haven't ruined the front: $36x^2$ is still nice—it is $6x$ times $6x$—and we have $-84x$ in the middle, and it is an even term.

Let's draw the square.

$36x^2$ comes from $6x$ times $6x$, and $-84x$ splits as $-42x$ and $-42x$.

$6x$ times something equals $-42x$. The something must be -7.

21.20

$$3x - 7x - 2 = 8$$
$$\times\ 12$$
$$36x^2 - 84x - 24 = 96$$
$$+\,73 \quad +\,73$$

	$6x$	-7
$6x$	$36x^2$	$-42x$
-7	$-42x$	$\boxed{49}$

This means that the magic number we seek is 49.

But we have -24. Add 24 and 49, or 73, to each side.

Now we have the equation $36x^2 - 84x + 49 = 169$.

The square shows us that this is really $(6x - 7)^2 = 169$ in disguise. And this we can solve!

$$36x^2 - 84x + 49 = 169$$
$$(6x - 7)^2 = 169$$
$$6x - 7^2 = 13 \text{ or } -13$$
$$6x = 20 \text{ or } -6$$
$$x = 10/3 \text{ or } -1$$

Our original quadratic has 2 solutions: $x = 3\,1/3$ or $x = -1$.

FURTHER EXPLORATION

WEB

Tanton, "Quadratics."
http://gdaymath.com/courses/quadratics/

PROBLEMS

1. Solve $3x^2 + 7x + 4 = 10$.

2. Use algebra to prove that the graph of $y = x^2 - 4x + 5$ matches the graph of $y = x^2$. (That is, that it is the same graph, just shifted vertically and horizontally.)

3. Solve $x^2 - 3x + 1 = 2$ first by multiplying through by 4 to avoid working with fractions, and then again without first multiplying by 4 and just dealing with fractions.

4. A riverboat travels upstream for 15 kilometers and then returns to the start position. The river has a steady current of 2 kilometers per hour, and the entire return journey took 5 hours. If the boat's speed was constant throughout the journey (that is, the speed of the boat relative to the water in which it moves is constant), for how long was the boat traveling upstream?

SOLUTIONS

21.21

	$6x$	7
$6x$	$36x^2$	$42x$
7	$42x$	$\boxed{49}$

1. Multiplying through by 3 and then by 4 gives $36x^2 + 84x + 48 = 120$. The box then suggests rewriting this as $36x^2 + 84x + 49 = 121$.

This is $(6x + 7)^2 = 121$, so $6x + 7 = 11$ or $6x + 7 = -11$. Thus, $6x$ equals 4 or -18, giving $x = 2/3$ or $x = -3$.

21.22

	x	-2
x	x^2	$-2x$
-2	$-2x$	$\boxed{4}$

2. Follow the box method for the equation $y = x^2 - 4x + 5$. The box suggests rewriting this as $y - 1 = x^2 - 4x + 4$, which is $y - 1 = (x - 2)^2$.

Thus, we have $y = (x - 2)^2 + 1$, which is the graph $y = x^2$ shifted horizontally 2 places and vertically 1 place.

21.23

	$2x$	-3
$2x$	$4x^2$	$-6x$
-3	$-6x$	$\boxed{9}$

3. Rewriting as $4x^2 - 12x + 4 = 8$, we get, from the box, $4x^2 - 12x + 9 = 13$.

Thus, we see $(2x - 3)^2 = 13$, giving

$$2x - 3 = \sqrt{13} \text{ or } -\sqrt{13}$$
$$2x = 3 + \sqrt{13} \text{ or } 3 - \sqrt{13}$$
$$2x = \frac{3}{2} + \frac{\sqrt{13}}{2} \text{ or } \frac{3}{2} - \frac{\sqrt{13}}{2}$$

21.24

	x	$\frac{-3}{2}$
x	x^2	$\frac{-3}{2}x$
$\frac{-3}{2}$	$\frac{-3}{2}x$	$\boxed{9/4}$

Alternatively, the box suggests rewriting $x^2 - 3x + 1 = 2$ as $x^2 - 3x + 9/4 = 13/4$.

Thus, we have $(x - 3/2)^2 = 13/4$, showing that $x - 3/2 = \sqrt{13}/2$ or $-\sqrt{13}/2$. This gives $x = 3/2 + \sqrt{13}/2$ or $3/2 - \sqrt{13}/2$, which is the same pair of answers as before.

4. Suppose that the boat's speed relative to the water is S kilometers per hour and that the boat traveled upstream for T hours (and hence downstream for $5 - T$ hours). With respect to the land, the boat was traveling at a speed of $S - 2$ kilometers per hour while going upstream and $S + 2$ kilometers per hour while going downstream.

Distance traveled is speed times time. So, for the upstream journey, we have $15 = (S - 2)T$.

For the downstream journey (also a 15-kilometer journey), we have $15 = (S + 2)(5 - T)$.

We seek the value of T. But we have Ss in these equations.

The first equation shows that $S - 2 = 15/T$, so $S = 15/T + 2$.

The second equation shows that $S + 2 = 15/(5 - T)$, so $S = 15/(5 - T) - 2$.

Now we can see that $15/T + 2 = 15/(5 - T) - 2$.

21.25

	$4T$	5
$4T$	$16T^2$	$20T$
5	$20T$	25

This is an (unfriendly) equation in just T. It can be rewritten as $15/T + 4 = 15/(5 - T)$. Multiply though by T and $5 - T$ to obviate fractions: $15(5 - T) + 4T(5 - T) = 15T$.

Expanding gives $75 - 15T + 20T - 4T^2 = 15T$, which is $4T^2 + 10T = 75$.

The box method gets us into fractions again (try it!), so multiply through by 4 again to avoid them: $16T^2 + 40T = 300$.

The box suggests rewriting this as $16T^2 + 40T + 25 = 325$.

We get $(4T + 5)^2 = 325$.

This yields the solutions $T = (\sqrt{325} - 5)/4 \approx 3.26$ and $T = (-\sqrt{325} - 5)/4 \approx -5.76$. Only the positive time value is relevant for this problem. Thus, the boat traveled upstream for approximately 3.26 hours.

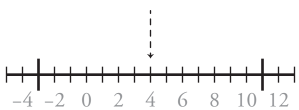
VISUALIZING BALANCE POINTS IN STATISTICS

LECTURE 22

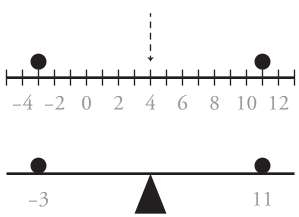
Quadratic equations arise in many applications. In this lecture, you will be exposed to an important application, one in the field of statistics. You will learn how solving a quadratic equation can be very helpful in looking at data and attempting to make meaningful predictions from it. First, you will look at data averages and find a surprising connection to physical balance points, and then quadratics will enter the scene by way of scatterplots.

DATA AVERAGES

22.1



22.2



What do we mean by the average of 2 numbers? The average of –3 and 11, for example, is the number in the middle of them, which is 4.

If we think of the 2 numbers as marbles, each weighing 1 gram, sitting on the number line, then the average of 2 numbers—their middle number—corresponds to the balance point between the 2 marbles. Think of 2 same-sized children sitting at the ends of a seesaw: Their balance point is right in the middle.

We have a formula for the balance point of 2 numbers x and y. Mathematically, their average is their sum divided by 2.

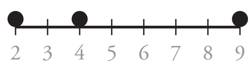
$$\text{Average of } x \text{ and } y = \frac{x + y}{2}$$

What about the average of 3 numbers, or of 8 numbers, or of 802 numbers? Are there multiple ways to visualize their averages, too?

The general formula for the average of any set of numbers is the sum of all the numbers divided by how many you have. For example, the average of 2, 4, and 9 is their sum, 2 + 4 + 9, or 15, divided by 3, giving 5.

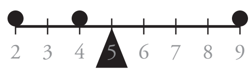
$$\text{Average of 2, 4, and 9} = \frac{2+4+9}{3} = 5$$

22.3



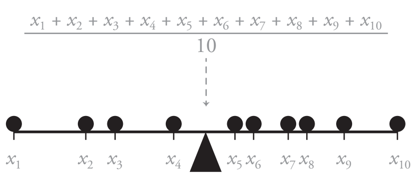
But what if we think of 2, 4, and 9 as points on the number line—or as 1-gram marbles positioned on the number line? Can we make sense of the average of these 3 numbers via balance points?

22.4



It turns out that the balance point of these 3 marbles is indeed at position 5, the average of the 3 numbers.

22.5



This is not obvious. In fact, the average of any number of numbers turns out to be the balance point of 1-gram marbles sitting at those positions. And that, too, is not obvious.

The idea that explains what is happening here is an ancient one. It is Archimedes's law of the lever.

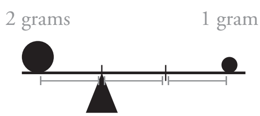
22.6

In the 3rd century B.C.E., Greek scholar Archimedes studied the placement of balance points between objects of equal and unequal weights.

Suppose that we have 2 marbles sitting on a rod like a seesaw, but one is twice as heavy as the other.

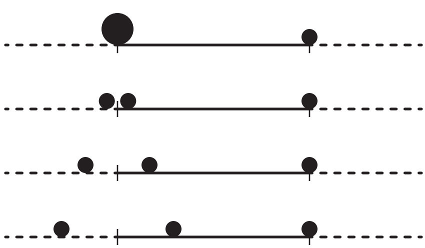
Intuitively, we expect the balance point for this system to be closer to the heavier marble than the lighter one. We might reason that because one marble is twice as heavy as the other, the balance point between them sits at a position with distances in a 2:1 ratio as well.

22.7



This turns out to be the case, and here's how Archimedes reasoned it so: Split the 2-gram marble into 2 1-gram marbles and place those new marbles equally spaced about the location of the original 2-gram piece. As far as balancing, those 2 new marbles are no different than a single 2-gram marble, and the balance point of the whole system has not changed.

22.8



In fact, the balance point of the system doesn't change if we space the 1-gram marbles differently. We just have to make sure that they are symmetrically placed about the original position of the 2-gram marble.

Let's space those 2 1-gram marbles out far enough to get an arrangement where all 3 marbles are equally spaced. We can get this spacing by dividing the original distance between marbles into thirds and placing the marbles as shown in **figure 22.9**.

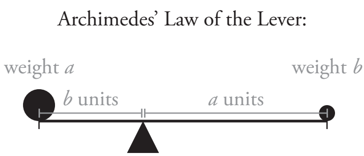
22.9

The balance point of 3 equally spaced 1-gram marbles is easy to identify. It is at the location of the middle marble. So, this is the balance point of the original system, too. And we see that it is indeed the location that divides in a 2:1 ratio.

22.10

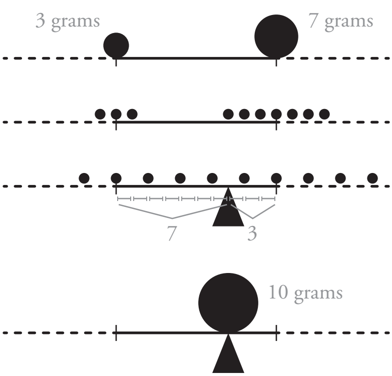
Archimedes reasoned that the balance point between 2 objects of different weights is always at the position that divides the distance between the objects in the same ratio as their weights. That's his law of the lever.

22.11



Physicists like thinking about the balance points of systems. They realize that 2 marbles sitting on a rod is a system that has the same physical behavior as a single marble sitting at the balance point. (And the weight of that single marble is the sum of the weights of the 2 original marbles.)

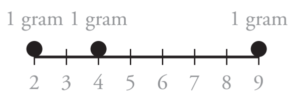
22.12



So, any 2 marbles sitting on the number line form a system that is physically equivalent to 1 heavy marble positioned at its balance point.

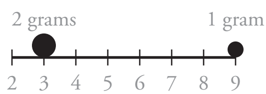
Let's make use of this. Let's go back to our 3 numbers 2, 4, and 9, which we saw had an average value 5, and work out the balance point of the system of 3 1-gram marbles at these positions.

22.13



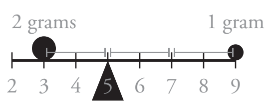
The 2 marbles at position 2 and 4 form their own subsystem. They have their own balance point at 3. So, these 2 1-gram marbles on the left form a system that is physically equivalent to a single 2-gram marble at position 3.

22.14



Where is the balance point of this new system? We have a 2-gram marble and a 1-gram marble. By the law of the lever, this has a balance point at a positon that divides the distance between them in a 2:1 ratio. We see that this is indeed at position 5.

22.15

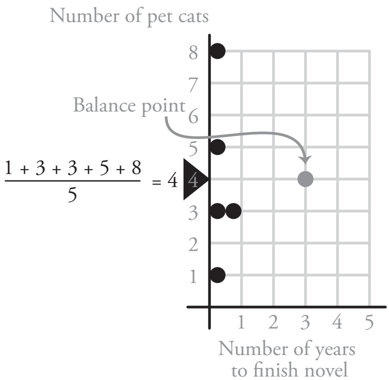


In this way, one can prove, by replacing pairs of marbles in turn and using the law of the lever over and over again, that the balance point of any collection of 1-gram marbles on the number line is indeed at the average value of those numbers. Averages are physical balance points.

In statistics, when you pinpoint the average of some data, also called the mean of the data, you are really pinpointing the balance point of the data just as though each data point is a 1-gram marble.

SCATTERPLOTS

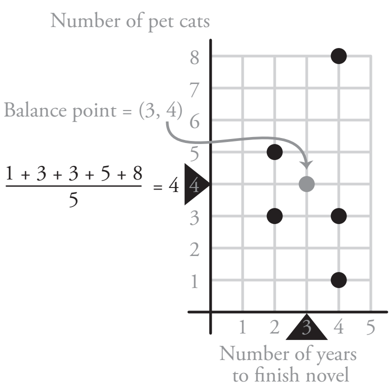
22.16



Here's an example of a 2-dimensional scatterplot in which 5 authors were each asked how many years it took them to write their first novel and how many pet cats they have at home.

In general, we can prove that the balance point of a 2-dimensional scatterplot is the point with a horizontal coordinate that is the average of all the horizontal data values and a vertical coordinate that is the average of all the vertical data values.

22.17



In general, when we collect data and display it, we are hoping to see some kind of trend. It's great if the data seems to follow a line, because we know how to find a formula for a line. And having a formula for data means that we would then have a mathematical understanding of the relationship between the variables we're studying and could make general predictions using the data.

Of course, not all data follows trends.

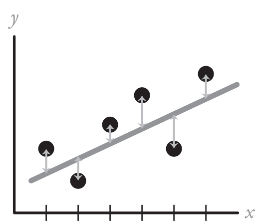
But the mathematical challenge for researchers who collect data and happen to see a linear trend in their data is the following: How do you choose the line that best represents what you see? Usually, there is a bit of wiggle room in the data. So, how do you know which line to use to best represent the trend that you have?

If we are looking for a quick answer to this, we can always set down a ruler on your graph and eyeball a line that looks like it fits pretty well. We can then work out the equation of the line from the picture.

But in a more precise study, we usually want a definite technique for finding the line of best possible fit. What might that technique be?

To consider this question, let's collect some general mathematical pieces. For example, the equation of a line is given by the formula $y = mx + b$, where m is the slope of the line and b is the value at which it crosses the y-axis.

22.18



Any deviation of a data point from a line must be in the vertical direction, not the horizontal direction. So, in looking for a line of best fit for a set of points on a scatterplot, any deviations we measure should be measured vertically.

The line of best fit should then be the line that gives us the smallest total deviation from that line. It is unlikely that we will have a line that fits perfectly. But we can look for one with the least total error.

For any number of data values, minimizing the sum of deviations from some line $y = mx + b$ always boils down to finding a minimal value of a quadratic expression for a value m. We can find these minimal values by doing the algebra of quadratics.

This approach to finding a line of best fit in statistics is called the least squares method because we look for a minimum to a sum of squares. It is a technique that has been used for the last 3 centuries.

The great German mathematician Carl Friedrich Gauss claims to have invented the method in 1795 when examining astronomical data but didn't publish the mathematics. In the early 1800s, other mathematicians and scientists developed the technique, too, and it has certainly been popular ever since in all sciences for line fitting.

FURTHER EXPLORATION

WEB

Tanton, "Averages as Fractions Added Wrong."
http://www.jamestanton.com/wp-content/uploads/2012/03/Curriculum-Essay_February-2016_Averages-as-Fractions.pdf

———, "Line of Best Fit: Least Squares Method."
http://www.jamestanton.com/?p=1175

READING

Stigler, *The History of Statistics*.

Tanton, *Thinking Mathematics! Vol. 8*.

PROBLEMS

1. A 2-gram and a 3-gram marble sit on a line.

Draw a sequence of diagrams to show that the balance point of this 2-marble system lies at a point ⅗ along the distance between the 2 marbles. (Follow Archimedes's approach: Replace the first marble with a pair of 1-gram marbles and the second with a triple of 1-gram marbles spaced so that the location of the balance point is unchanged.)

2. Four 1-gram marbles are placed at positions a, b, c, and d along the number line. Use Archimedes's law of the lever to prove that the balance point of this system is at position $(a + b + c + d)/4$.

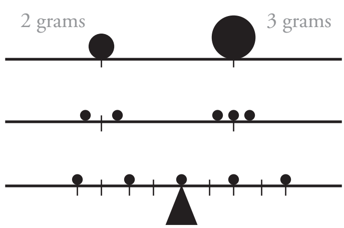
3. Using the least squares method, find the line of best fit for the 3 data points (1, 1), (3, 4), and (5, 4).

4. One-gram marbles are placed at positions (x_1, y_1), (x_2, y_2), … (x_n, y_n), in the plane. Explain why the balance point of this system lies at $(\bar{x}, \bar{y})$, where $\bar{x} = (x_1 + x_2 + \ldots + x_n)/n$ is the average of the x-values and $\bar{y} = (y_1 + y_2 + \ldots + y_n)/n$ is the average of the y-values.

SOLUTIONS

22.19

1.



2. The two 1-gram marbles at positions a and b are equivalent to a single 2-gram marble sitting at the position halfway between them, $(a + b)/2$. Similarly, the two 1-gram marbles at positions c and d are equivalent to a single 2-gram marble sitting at the position halfway between them, $(c + d)/2$.

These two 2-gram marbles are equivalent to a single 4-gram marble sitting halfway between them. This is the position

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2}.$$

This can be rewritten as

$$\frac{1}{2}\left(\frac{a+b}{2} + \frac{c+d}{2}\right) = \frac{1}{2}\left(\frac{a+b+c+d}{2}\right) = \frac{a+b+c+d}{4}.$$

3. The average value of the horizontal data values is $(1 + 3 + 5)/3 = 3$, and the average value of the vertical data values is $(1 + 4 + 4)/3 = 3$. The line of best fit thus passes through the point $(3, 3)$.

Putting $x = 3$ and $y = 3$ into the general equation of a line $y = mx + b$ gives $3 = 3m + b$, so $b = 3 - 3m$, so we need to work with the equation $y = mx + 3 - 3m$.

When $x = 1$, the line gives $y = 3 - 2m$, but the data wants $y = 1$. These values differ by $(3 - 2m) - 1 = 2 - 2m$.

When $x = 3$, the line gives $y = 3$, but the data wants $y = 4$. These values differ by 1.

When $x = 5$, the line gives $y = 2m + 3$, but the data wants $y = 4$. These values differ by $(2m + 3) - 4 = 2m - 1$.

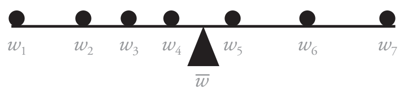
The sum of differences squared is $(2 - 2m)^2 + 1^2 + (2m - 1)^2$. This expands (draw boxes) to become $(4 + 4m^2 - 8m) + 1 + (4m^2 + 1 - 4m) = 8m^2 - 12m + 6$. We want to find the value of m that makes this as small as possible.

Rewriting $8m^2 - 12m + 6$ as $4m(2m - 3) + 6$, we see that line of symmetry of this quadratic is halfway between $m = 0$ and $m = 3/2$. Thus, the value of m we seek is $m = 3/4$.

The line of best fit, $y = mx + 3 - 3m$, is thus $y = (3/4)x + 3 - 9/4 = (3/4)x + 3/4$.

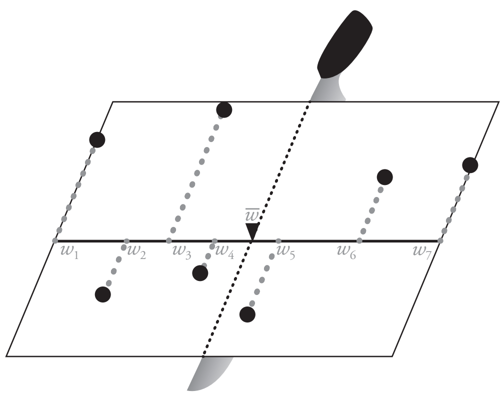
4. If a set of 1-gram marbles are placed on a line at positions $w_1, w_2, \ldots, w_n$, then we know that the balance point of that system lies at position $\overline{w} = (w_1 + w_2 + \ldots + w_n)/n$.

22.20

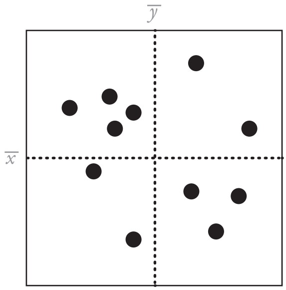


If we slide these marbles in a plane in a direction that is orthogonal to the line, then the 2-dimensional system of marbles balances along a knife's edge placed at position $\overline{w}$ that is orthogonal to the line.

22.21



22.22



Now consider the system described in the question. The marbles there balance along a knife's edge along a vertical line at position $\overline{x}$. By the same token, the marbles also balance along a knife's edge along a horizontal line at position $\overline{y}$.

Thus, the entire system balances at the single point $(\overline{x}, \overline{y})$, where the 2 knife's edges cross.

VISUALIZING FIXED POINTS

LECTURE 23

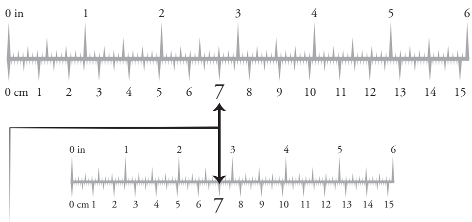
The study of fixed points is a big topic in mathematics. If you have a system that is constantly rearranging itself, you might want to know if there are any stable features about it. Economists, in particular, are very interested in fixed-point theories. Many of the economic models they devise can be seen as turbulent, self-adjusting systems, and economists want to be sure that stable, invariant prices are nonetheless sure to exist within the system—that is, they want a mathematical proof that a fixed point exists. This lecture will prove the fixed-point theorem.

FIXED POINTS

Let's say that we have 2 sheets of paper and that we lie one on top of the other, with every point aligned. If we were to crumple one of the sheets of paper and place it on top of the other piece of paper, will there be a matching fixed point?

In placing a scaled ruler next to a copy of its original self, we're always sure to find a matching fixed point.

23.1



7 is a matching point

The paper question is a 2-dimensional version of this: Instead of shrinking the paper, we're crumpling it, and that seems much more complicated. Nevertheless, must there be a fixed matching point for the crumpled paper? Do you think that there is sure to be a fixed point every time, or do you think that there is too much potential variability to guarantee a fixed point each time?

In this example, which involves crumpling one piece of paper and placing it on top of the other, a fixed point is indeed sure to exist. No matter how crumpled the paper and no matter how you place that crumpled paper on top, there is certain to be at least 1 matching point between the 2 papers. We're not saying that the fixed point is actually fixed in any sense—move the paper around and you'll get a different fixed point. But no matter where you place the paper, at least 1 fixed point is absolutely sure to exist.

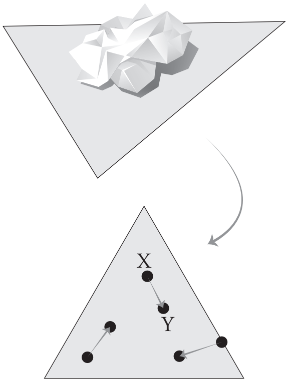
There are 2 caveats to this. First, we must keep the crumpled paper on top of the bottom sheet. There are clearly no fixed points if we have the sheets completely misaligned. Second, we are assuming that we don't tear the paper. We want to maintain the continuous paper surface. If we tear the paper, all bets are off!

TRIANGULAR PAPER

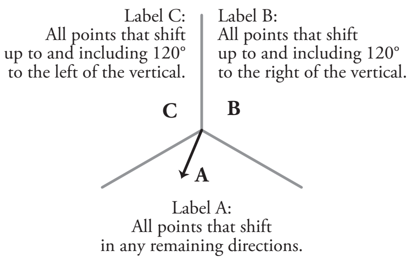
Let's prove our paper-crumpling claim, but let's first prove it for triangular paper.

Let's say that we have 2 sheets of paper, the first initially lying on top of the other. If we crumple the top paper and put it back on the bottom sheet, then we claim that there is at least 1 point in the crumpled paper sitting directly above its matching point on the bottom sheet.

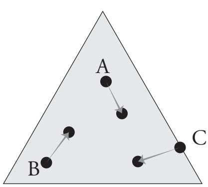
23.2



23.3



23.4



Each point of the crumpled paper is likely to be sitting above a point that is different from its original uncrumpled location. Rather than draw a picture of 2 sheets of paper, one crumpled and the other not, we can draw a picture of a single uncrumpled triangle. On that picture, we can show where each point of the triangle moves when the paper is crumpled.

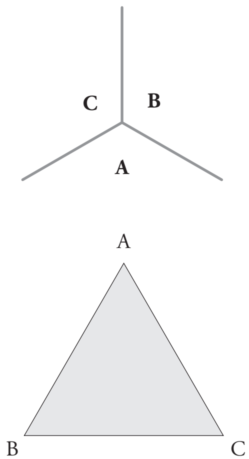
For example, **figure 23.2** shows that the point X in the triangle moves to position Y in the crumpling process. In this way, we have all of the information about the crumpling encoded in a picture of a single triangle.

Let's give each point of the triangle a label—either A, B, or C—according to the direction it moves.

If a point moves mostly downward, label it A. If it moves mostly northeast, label it B. If it moves mostly northwest, label it C. **Figure 23.3** illustrates the general labeling rule.

For example, in our picture of 3 points being shifted, one is labeled A because it shifts downward, one is labeled B because it shifts eastward, and one is labeled C because it shifts westward.

23.5

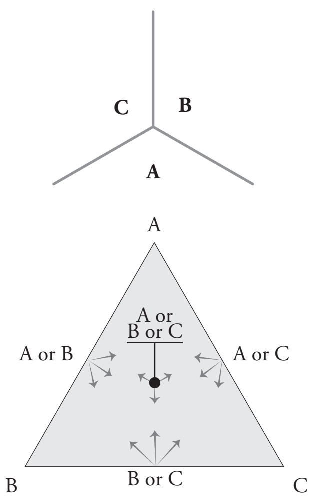


The only points we can't label are the ones that don't move at all. But if we find such a point, then we have what we are looking for—a fixed point, a point on the crumple sitting directly above its matching point on the bottom sheet of paper.

The worst-case scenario, then, would be if we never come across a point that doesn't move. So, let's assume that we are in this worst-case scenario—that we never stumble across a fixed point in what we are about to do. Let's see what we can deduce nonetheless.

The point at the top of the triangle can only move downward; thus, it must have label A. The bottom-left corner of the triangle can only move northeast; it must have label B. Similarly, the third vertex of the triangle must have label C.

23.6

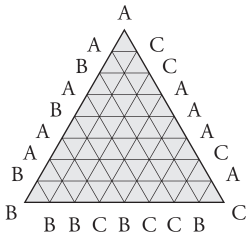


A point on the left edge of the triangle can only move northeast or possibly downward. So, each such point on the left edge must be labeled A or B.

Similarly, any point on the edge with vertices B and C, the bottom edge, will be labeled B or C (none of those points can possibly move downward, so they must move in the B or C direction).

Any point on the edge with vertices A and C, the right edge, will be labeled A or C. An interior point of the triangle can be labeled either A, B, or C, because it could move in any direction.

23.7



Every point of the triangle has a label according to how it shifts in crumpling.

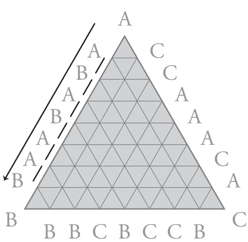
Next, let's apply this labeling to a palace diagram. Let's draw a palace diagram on the triangle as shown in **figure 23.7**.

We have subdivided our triangle into a large number of smaller triangles, following a regular pattern. Let's say that each small triangle has an area of less than 0.1 square units.

Each point in this palace diagram is labeled either A, B, or C. We've just shown that all the labels on the left edge must be either A or B. We also have just Bs and Cs on the bottom edge and just As and Cs on the right edge.

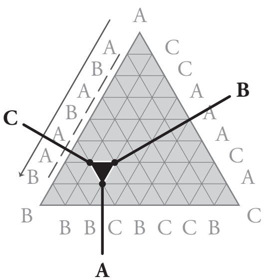
We're interested in outside AB doors, and we see that all the outside AB doors appear just on the left edge of the big triangle.

23.8



Notice that the number of outside AB doors must be odd. Imagine starting at the top point labeled A and walking down the left edge to the bottom corner B. Each time you encounter a change of letter—from A to B, or B to A—you've just traversed an outside AB door. But because we start with an A and end with a B, we must encounter a change of letter an odd number of times. Hence, we must have traversed an odd number of outside AB doors along this edge.

23.9



So, our palace diagram always has an odd number of outside AB doors. There is at least 1 fully labeled ABC triangle somewhere in this diagram. And this ABC triangle has an area of no more than 0.1 square units.

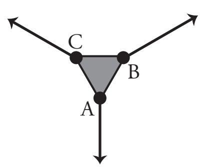
Actually, in the same way, by using a finer triangulation, we can find an ABC triangle with an area no larger than 0.01 square units, or another with an area no larger than 0.001 square units, and so on.

In this way, we can find an infinite sequence of fully labeled ABC triangles of areas decreasing by factors of 10.

We can now finish off the proof with an intuitive argument.

Pick one of these fully labeled ABC triangles, one smaller than the size of an atom. Think about what is happening to its 3 corners: one corner moves northeast, one corner moves northwest, and one corner moves downward. The 3 corners of this absurdly small triangle are moving in 3 different directions—and we're not tearing the paper!

23.10



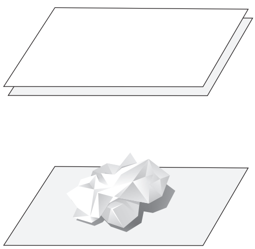
This surely can't hold for smaller and smaller triangles—triangles smaller than the smallest of subatomic particles. The only way out of this philosophical pickle is for this tiniest of tiniest triangles to be sitting around a point that doesn't move at all—that is, we must be honing in on a fixed point in the big triangle.

So, even in the worst-case scenario of us not stumbling across a fixed point by pure luck, we're concluding that there must be a point that doesn't move nonetheless. We have thus proved that crumpling paper must have at least 1 fixed point, no matter what.

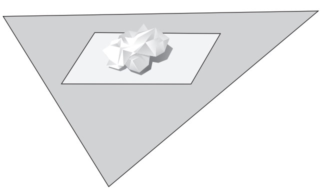
There is a way to take this idea of looking at a sequence of smaller and smaller fully labeled triangles and rigorously prove that a fixed-limit point really does exist.

RECTANGULAR PAPER

23.11



23.12



So, we have proved our crumpled-paper fixed-point theorem—for triangular paper. What about rectangular paper?

It must work for rectangular paper, too. All we need to do is imagine rectangular sheets glued onto large triangular sheets. Crumple the large triangle and place that crumpled paper somewhere on the rectangle of the bottom sheet. We know that there must be a fixed point for the triangles.

But because the crumpled paper is sitting on top of the rectangle, that fixed point must actually be within the rectangles—just as we were hoping to see.

We've established what mathematician's call the Brouwer fixed-point theorem, at least, in dimension 2. Luitzen Brouwer was a Dutch mathematician who lived from 1881 to 1966. His theorem says that any continuous map from a bounded, simply connected region in the plane to itself—that is, any continuous motion within a square or a triangle, for example, a shape with no holes—must possess at least 1 fixed point. In other words, there is sure to be at least 1 point that does not move under the motion.

The theorem actually works in all dimensions. Some people like to interpret the 3-dimensional version of the theorem as follows: Stir a cup of coffee. After stirring, once the liquid has settled back down, at least 1 point of the liquid is sure to have returned to its original location.

Actually, there are problems with this interpretation. For example, liquid coffee is composed of discrete molecules that constantly jitter, and the spoon for stirring might separate points, causing the mixing not to be continuous. Nonetheless, this coffee analogy captures the essence of the 3-dimensional theorem pretty well.

Brouwer didn't prove his fixed-point theorem the way we did. The proof we went through is due to 20th-century German mathematician Emanuel Sperner. The result about labeling palace diagrams is today known as Sperner's lemma.

FURTHER EXPLORATION

READING

Border, *Fixed Point Theorems with Applications to Economics and Game Theory*.

Jarvis and Tanton, "The Hairy Ball Theorem via Sperner's Lemma." (Learn how Sperner's lemma can be used to prove the hairy ball theorem in mathematics.)

Kac and Ulam, *Mathematics and Logic*.

PROBLEMS

1. Imagine 2 sheets of paper, one lying directly on top of the other. At present, each point on the top sheet lies directly above its matching point on the bottom sheet. The Brouwer fixed-point theorem says that if we crumple the top sheet and place it back on top of the bottom sheet, there is sure to still be at least 1 point in the crumpled paper sitting directly above its matching point in the bottom sheet—provided that the paper is not torn in the crumpling process.

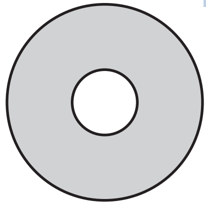
Give an example to show that tearing the top sheet can indeed lead to a situation in which no fixed point exists.

2. Lulu draws dots all over the surface of a rubber ball and randomly labels each of the dots either A, B, or C. She then draws lines connecting the dots so that the whole surface of the ball is then covered with (curved) triangles, each triangle having 3 edges and just 3 labeled dots at its corners.

Lulu notices that 1 of the triangles she formed is fully labeled: It has 1 corner dot labeled A, 1 labeled B, and 1 labeled C. Explain why, if Lulu were to look further, she would be sure to find a second fully labeled triangle on the ball's surface.

3. Brouwer's fixed-point theorem does not apply to shapes with holes. Show that there is a way to map points of an annulus (a ring shape) to itself and never have a fixed point.

23.13



SOLUTIONS

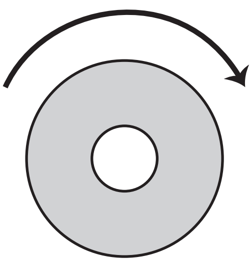
1. Simply tear the top sheet in half and switch the places of the 2 halves. There is no fixed point.

2. Imagine the triangles as the floor plan of a palace with triangular rooms on the surface of the ball and the edges labeled AB as doors between rooms. Walk a path from the fully labeled ABC triangle as far as you can through the AB doors without backtracking through a door. The path must eventually halt, and it must be by entering a room that has only 1 AB door. This final room is a second fully labeled ABC triangle.

This shows that all fully labeled triangles drawn on the surface of a ball come in pairs. In particular, there cannot be just 1 such triangle.

3. The mapping that rotates each point in the annulus 10° clockwise around the center of the ring has no fixed points. (If the annulus were a full disk with no central hole, then there would be a fixed point for this mapping—namely, the center point.)

23.14



BRINGING VISUAL MATHEMATICS TOGETHER

LECTURE 24

At this point, in this final lecture of the course, you have a whole array of visual delights, devices, and powerful mathematical ideas at your mental grasp, including base machines, folding, fractions, fixed points, infinity, and more. This lecture will return to the patterns of folding and show a level of deep arithmetic that brings together so many of the ideas that have been explored in this course—and pushes them even further.

FOLDING INSTRUCTIONS AND BINARY REPRESENTATIONS

24.1

$\frac{1}{2} \longleftrightarrow$ R

$\frac{1}{4} \longleftrightarrow$ RL

$\frac{5}{8} \longleftrightarrow$ RLR

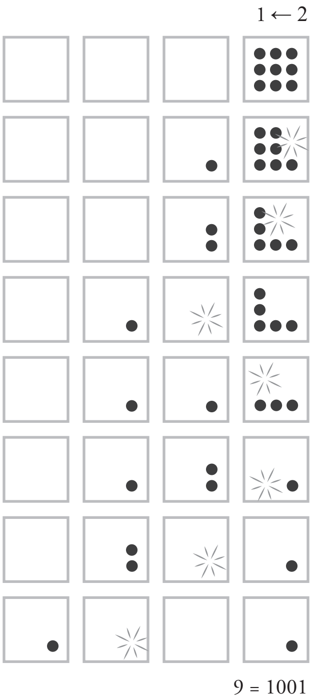
$\frac{13}{16} \longleftrightarrow$ RLRR

$\frac{13}{32} \longleftrightarrow$ RLRRL

Let's say that we have a strip of paper that is 1 unit long. Instructions for folding specific fractions are as follows. To fold the fraction $\frac{1}{4}$, for example, pick up the right end and fold it over, and then pick up the left end and fold it to the crease you just made. We can shorten these instructions with the note "right, left," or RL. To fold the fraction $\frac{5}{8}$, follow the instructions: RLR, which means right fold, left fold, right fold.

These instructions go back to base machines—in particular, the $1 \leftarrow 2$ machine. Recall how this works: We have a row of boxes and we put in dots. Dots always go in the rightmost box, and whenever there are 2 dots in a box, they explode and are replaced by 1 dot that is 1 place to the left. For example, putting 9 dots in the machine eventually gives us the code 1001 for the number 9 in base 2. (See **figure 24.2**.)

24.2



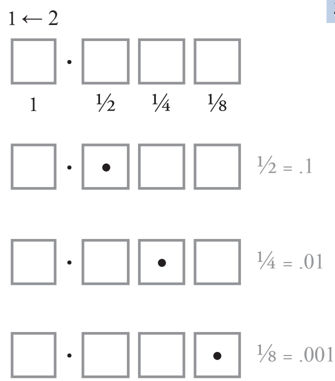
But we want to talk about fractions, and to get those we need to look at decimals in a base 2—though "decimals" ("deci-") is not the right word. Perhaps "bimals" is better. In any case, we want a decimal point—a bimal point—and some boxes to the right. Just as decimals in base 10 work with tenths, hundredths, thousandths, and so on, the powers of 1⁄10, bimals, work with the powers of ½: a half, a quarter, an eighth, and so on. (See **figure 24.3**.)

Let's work out some bimals—that is, the binary representations of some fractions. What's ½?

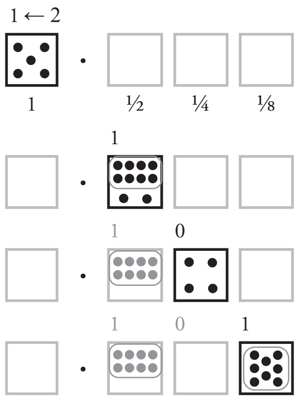
It's 1 dot in the ½ box: .1.

¼ is 1 dot in the quarter box: .01.

24.3



24.4



⅝ = .101

What is the binary representation of ⅝ in base 2? This is not so obvious. We can think of ⅝ as a division problem: 5 divided by 8. So, here's 5 dots, and let's look for groups of 8 in this picture—that is, let's do the division.

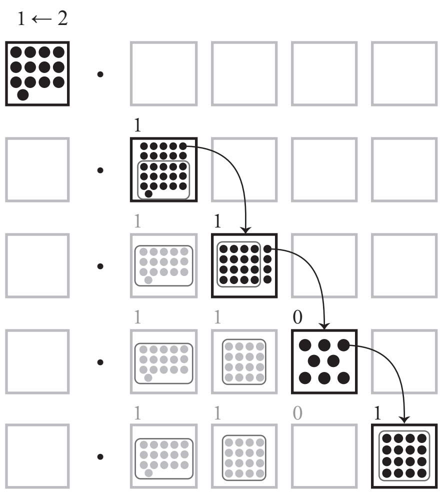
Unexplode the 5 dots to see 10 dots in the halves place. We see 1 group of 8 there.

But there are 2 dots left over. Unexploding them once gives 4 dots.

Unexploding them, too, gives us another group of 8. Then, all of the dots are accounted for.

The binary representation of ⅝ is .101.

24.5



13⁄16 = .1101

For 13⁄16, we can write numbers instead of drawing dots.

First, we have 13 dots in a box. They all unexplode to make 26 dots in a neighboring box. There's 1 group of 16 among them, leaving 10 behind to deal with. Those 10 unexplode to make 20 in the next box. There's a group of 16 among those 20, leaving 4 behind to still attend to. Those 4 unexplode twice to reveal 1 final group of 16.

We see that 13⁄16 = .1101.

24.6

$$\frac{1}{2} \longleftrightarrow \text{R} \qquad \frac{1}{2} \longleftrightarrow .1$$

$$\frac{1}{4} \longleftrightarrow \text{RL} \qquad \frac{1}{4} \longleftrightarrow .01$$

$$\frac{5}{8} \longleftrightarrow \text{RLR} \qquad \frac{5}{8} \longleftrightarrow .101$$

$$\frac{13}{16} \longleftrightarrow \text{RLRR} \qquad \frac{13}{16} \longleftrightarrow .1101$$

$$\frac{13}{32} \longleftrightarrow \text{RLRRL} \qquad \frac{13}{32} \longleftrightarrow .01101$$

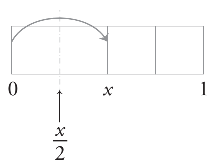
Let's go back to the folding codes and compare them with the binary representations we've computed.

The binary representation of $13/32$ turns out to be .01101.

Equate ones with Rs and zeroes with Ls and read the binary representations backward. For example, $13/16$ has binary representation .1101. This is 1011 backward, which as Ls and Rs reads RLRR—the exact folding instructions!

$13/32$ is .01101. This is 10110 backward. As Ls and Rs, this is RLRRL.

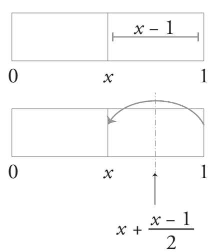
24.7



What's the connection between folding and binary?

To answer that question, let's look at what happens when you add a left fold or a right fold to a preexisting crease.

24.8



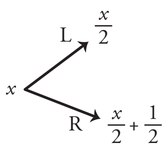
Suppose that we have a crease at position x. If we perform a left fold, we get a new crease at position $x/2$.

Now suppose that we perform a right fold instead. This divides the right portion of the strip in half. The right portion is $1 - x$ units long. So, we get a new crease that is x units plus half of $1 - x$ units along the strip.

Some algebra shows that this new crease is at position $x/2 + 1/2$.

$$x + \frac{1-x}{2} = x + \frac{1}{2} - \frac{x}{2} = \frac{x}{2} + \frac{1}{2}$$

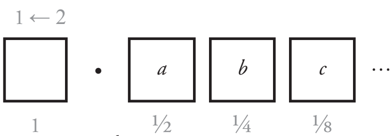
24.9



In summary, if we have a crease at positon x, then a left fold makes a new crease at position $x/2$, half the distance, and a right fold makes a new crease at position $x/2 + 1/2$.

In binary, x is "point something something something." Let's write x as $.abc\ldots$, where the a, b, and c, etc., are each either 1 or 0. This means that x is a halves (either 1 or 0 of them), b quarters, c eighths, and so on.

24.10



$x = .abc\ldots$

$$x = \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \ldots$$

÷ 2

$$\frac{x}{2} = \frac{a}{4} + \frac{b}{8} + \frac{c}{16} + \ldots$$

$$\frac{x}{2} = .0abc\ldots$$

A left fold turns x into $x/2$. Dividing everything by 2, we see that $x/2$ is a quarters, b eighths, and c sixteenths, and so on—that is, $x/2$ is $.0abc\ldots$ in binary.

So, a left fold has the effect of inserting a 0 in the front of the binary representation of x.

A right fold, on the other hand, turns x into $x/2 + 1/2$. We see that this has the effect of inserting 1 into the binary representation of x: We saw that $x/2$ is a quarters, b eighths, and c sixteenths, and so on, so $x/2 + 1/2$ is $1/2$ and a quarters, b eighths, and c sixteens, etc. We have $.1abc\ldots$.

$$\frac{x}{2} + \frac{1}{2} = \frac{1}{2} + \frac{a}{4} + \frac{b}{8} + \frac{c}{16} + \ldots$$

$$\frac{x}{2} + \frac{1}{2} = .1abc\ldots$$

We see that ones really do correspond to right folds and zeros really do correspond to left folds. But because we are always inserting ones and zeroes at the beginning of binary representations, we're writing things down in the reverse order we expect. For example, let's fold 13⁄16 again. It has binary representation .1101. How do we get that through folding?

24.11

$$\frac{13}{16} = .1101 \qquad \frac{13}{16} \longleftrightarrow \text{RLRR}$$

To construct .1101, start with nothing, 0, and first insert a 1 to get .1. That's a right fold.

24.12

0	
.1	First R
.01	Then L
.101	Then R
.1101	Then R

$$\frac{13}{16} = .1101 \qquad \frac{13}{16} \longleftrightarrow \text{RLRR}$$

Then, insert a 0 to get .01. That's a left fold.

Then, insert a 1 to get .101. That's a right fold.

Then, insert a 1 to get .1101. That's a right fold.

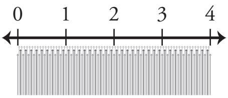
So, RLRR creates .1101. That's 13⁄16.

The binary representations match folding instructions, except in reverse order.

We can fold any fraction with a power of 2 in its denominator. All we have to do is work out the binary representation of that number in a 1 ← 2 machine, and then read the ones and zeroes as instructions: 1 is a right fold and 0 is a left fold.

FOLDING FRACTIONS

24.13



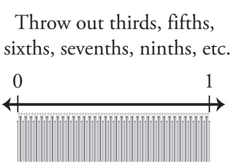
We can position a folding fraction—that is, a fraction with a denominator that is a power of 2—as close as we like to any real-number position on the strip. This observation actually takes the results of lecture 12 a bit further.

Recall that all fractions seem to densely fill up the number line. Despite this, the set of all fractions takes up no space on the number line.

In our folding game, we're only looking at fractions in the interval from 0 to 1. Moreover, all the fractions we create from folding involve the powers of 2: They're halves, quarters, eighths, sixteenths, and so on. So, in folding, we're only playing with a small subclass of fractions—those that have a denominator that is a power of 2. Yet we've showed that we can get these folding fractions positioned arbitrarily closely to all possible positions in the interval from 0 to 1.

From lecture 12, we know that all of the fractions between 0 and 1 densely fill up that interval.

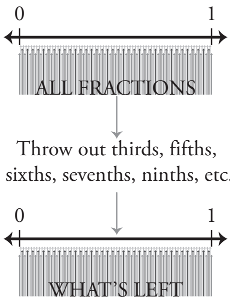
24.14



Now throw out all the fractions that have denominators that are not a power of 2—all the thirds (⅓, ⅔, 3⁄3, …), all the fifths (⅕, ⅖, ⅗, …), all the sixths, all the sevenths, all the ninths, all the tenths, all the twentieths, and so on. Throw out what has to be basically all the fractions.

What's left are just the folding fractions—the fractions with a denominator that is a power of 2. And we just saw that these fractions still densely fill up the interval from 0 to 1.

24.15



In other words, throw out all of those infinitudes of fractions of different types and we will be left with a picture that looks exactly the same as it did before!

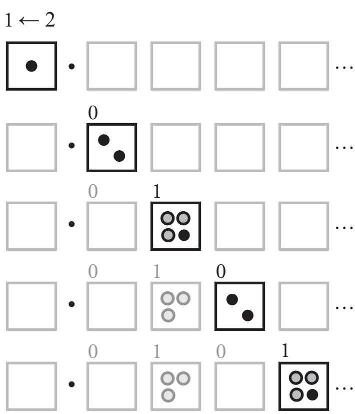
Infinity is just counterintuitive and weird!

But here's a weirder thing about infinity. We can often play with it and use it to our advantage.

Recall from lecture 10 that all fractions have repeating decimal expansions. This is true in base 2 as well. For example, in base 2, the fraction ⅓ has infinite repeating bimal .01010101….

In fact, we can quickly do the division to see that this is right.

24.16



⅓ = .01010101…

Here's 1 dot in a 1 ← 2 machine. We're looking for groups of 3.

Unexplode once, and then again to see a group of 3 finally.

But we have 1 dot left over. Unexplode that dot once, and then again to see another group of 3.

But again we have 1 dot left over. Unexplode twice again to find another group of 3 and 1 dot left over. We're in an infinite process, and we see that ⅓ indeed has an infinite binary representation: .01010101….

Now, let's use the fact that this representation is infinite to our advantage.

$\frac{1}{3}$ = .01010101…. Notice that inserting a 1 and then inserting a 0 into the expansion doesn't change the expansion.

This is true only because the expansion is infinitely long. Adding 2 more decimal places to a number usually changes the number.

So, following the language of lecture 23, we see that the number $\frac{1}{3}$ is a fixed point of the operation of inserting a 1 and then a 0. In folding, inserting a 1 corresponds to a right fold, and inserting a 0 corresponds to a left fold. This means that the number $\frac{1}{3}$ must be a fixed point in a fold right, fold left pattern—and it turns out that it is!

FURTHER EXPLORATION

READING

Iga, "The Truck Driver's Straw Problem and Cantor Sets."
Tanton, *Mathematics Galore!*

PROBLEMS

1. Given a strip of paper, provide a set of folding instructions that will produce a crease mark at the position that is $\frac{7}{8}$ of the way along the strip. (Start with a right fold to make a crease at position $\frac{1}{2}$. Then, each fold thereafter, a left fold or a right fold, is a fold to the mark previously just made.) Also provide instructions for producing a fold at position $\frac{21}{32}$.

2. Following the idea of question 1, give a set of folding instructions that will produce a crease very near the position that is $\frac{3}{7}$ of the way along the strip.

3. You have 2 piles of sand: a left pile and a right pile. Performing a "left move" means transferring $\frac{2}{3}$ of the sand in the left pile over to the right pile. A "right move" means transferring $\frac{2}{3}$ of the sand from the right pile over to the left.

If you were to alternately perform left and right moves, what would you see in the long run?

4. Following question 3, suppose that the left pile of sand has all the sand and the right pile of sand is empty. Is there a sequence of left and right moves, in any order, that will lead to 2 equal piles of sand? (The first meaningful move you can make is a left move, producing a left pile with $\frac{1}{3}$ of the sand and right pile with $\frac{2}{3}$ of the sand. The next move could either be another left move or a right move.)

SOLUTIONS

1. We have $\frac{7}{8} = (4 + 2 + 1)/8 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$. This has representation .111 in base 2. Thus, a series of 3 right folds will produce a crease at position $\frac{7}{8}$.

We have $\frac{21}{32} = (16 + 4 + 1)/32 = \frac{1}{2} + \frac{1}{8} + \frac{1}{32}$. This has representation .10101 in base 2. Thus, the instructions RLRLR (right fold, left fold, right fold, left fold, right fold) will produce a crease at position $\frac{21}{32}$.

2. Performing the division $3 \div 7$ in a $1 \leftarrow 2$ machine shows that $\frac{3}{7}$ has representation $.011011011\ldots = .\overline{011}$ in base 2.

Thus, performing the sequence of instructions RRL (right fold, right fold, left fold) multiple times will produce a crease mark as close to the $\frac{3}{7}$ mark as desired.

3. Suppose that we have 1 unit of sand in total (1 pound, 1 gallon, or 1 cup, for example). If the left pile has x units of sand, then the right pile has $1 - x$ units.

A left move changes the amount of sand in the left pile from x to $x/3$.

A right move changes the amount of sand in the left pile from x to $x + (2/3)(1 - x) = 2/3 + x/3$.

Let's write x as a "decimal" in base 3: $x = .abc\ldots$ Then, a left move changes $.abc\ldots$ to $.0abc\ldots$, and a right move changes $.abc\ldots$ to $.2abc\ldots$. (So, a left move inserts a 0, and a right move inserts a 2.)

Now suppose that we repeatedly perform left and right moves. The amount of sand in the left pile changes as follows:

$.abc$
$.0abc$
$.20abc$
$.020abc$
$.2020abc$
$.02020abc$
$.202020abc$
$\ldots$

The amount of sand in the left pile thus approaches a state that oscillates between the 2 values $.0202020\ldots = .\overline{02}$ and $.2020202\ldots = .\overline{20}$. What are these values?

Let $F = .0202020\ldots$. Because this is written in base 3, multiply by 3 and then 3 again:

$3F = .20202020\ldots$
$9F = 2.0202020\ldots = 2 + F$

We see that $8F = 2$, so $F = 1/4$. Similarly, $.2020202\ldots = 3/4$.

In the long run, repeatedly performing alternating left and right moves has the system approach an oscillatory state of $1/4$ and $3/4$ units of sand for the 2 piles.

4. Performing the division $1 \div 2$ in a $1 \leftarrow 3$ machine shows that, in base 3, ½ has representation .11111….

As we saw in question 3, performing a left move or a right move inserts either a 0 or a 2 into the base-3 representation of a pile amount. It is impossible to produce .11111…. We will never see 2 equal piles of sand.

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